2-1 $\underline{w} = \underline{\mathcal{E}} \times \underline{\mathcal{H}} = \mathrm{Re}[\underline{E}e^{j\omega t}] \times \mathrm{Re}[\underline{H}e^{j\omega t}]$

Using the identity $\mathrm{Re}[\underline{A}e^{j\omega t}] = \frac{1}{2}[\underline{E}e^{j\omega t} + \underline{E}^* e^{-j\omega t}]$

The instant Poynting vector can be written as

$$\underline{w} = \left\{\frac{1}{2}[\underline{E}e^{j\omega t} + \underline{E}^* e^{-j\omega t}]\right\} \times \left\{\frac{1}{2}[\underline{H}e^{j\omega t} + \underline{H}^* e^{-j\omega t}]\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2}[\underline{E}\times\underline{H}^* + \underline{E}^*\times\underline{H}] + \frac{1}{2}[\underline{E}\times\underline{H}e^{j2\omega t} + \underline{E}^*\times\underline{H}^* e^{-j\omega t}]\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2}[\underline{E}\times\underline{H}^* + (\underline{E}\times\underline{H}^*)^*] + \frac{1}{2}[\underline{E}\times\underline{H}e^{j2\omega t} + (\underline{E}\times\underline{H}e^{j\omega t})^*]\right\}$$

Using the above identity again, but this time in reverse order, we can write that

$$\underline{w} = \frac{1}{2}[\mathrm{Re}(E\times H^*)] + \frac{1}{2}[\mathrm{Re}(\underline{E}\times\underline{H}e^{j2\omega t})]$$

2-2. (a) $\underline{W}_{rad} = \frac{1}{2}[\underline{E}\times\underline{H}^*] = \frac{E^2}{2\eta}\hat{a}_r = \frac{5^2\hat{a}_r}{2(120\pi)} = 0.03315\,\hat{a}_r$ watts/m$^2$

(b) $P_{rad} = \oint_s W_{rad}\,dS = \int_0^{2\pi}\int_0^{\pi}(0.03315)(r^2\sin\theta\,d\theta\,d\phi)$

$$= \int_0^{2\pi}\int_0^{\pi}(0.03315)(100)^2\sin\theta\,d\theta\,d\phi$$

$$= 2\pi(0.03315)(100)^2\int_0^{\pi}\sin\theta\,d\theta = 2\pi(0.03315)(100)^2\cdot 2$$

$$= 4165.75 \text{ watts}$$

2-3. (a) $D_o = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi(200\times10^{-3})}{0.9(125.66\times10^{-3})} = 22.22 = 13.47\text{dB}$

$$G_o = e_r \cdot D_o = 0.9(22.22) = 20 = 13.01\text{dB}$$

(b) $D_o = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi(200\times10^{-3})}{(125.66\times10^{-3})} = 20 = 13.01\text{dB}$

$$G_o = e_r \cdot D_o = 0.9\cdot(20) = 18 = 12.55\text{dB}$$

2-4 $U = B_0 \cos^3\theta$

(a) $P_{rad} = \int_0^{2\pi}\int_0^{\pi/2} U \sin\theta \, d\theta \, d\phi = B_0 \int_0^{2\pi}\int_0^{\pi/2} \cos^3\theta \sin\theta \, d\theta \, d\phi$

$= 2\pi B_0 \int_0^{\pi/2} \cos^3\theta \sin\theta \, d\theta$

$P_{rad} = 2\pi B_0 \left(-\frac{\cos^4\theta}{4}\right)\Big|_0^{\pi/2} = \frac{\pi}{2} B_0 = 10 \Rightarrow B_0 = \frac{20}{\pi} = 6.3662$

$U = 6.3662 \cos^3\theta$

$W = \frac{U}{r^2} = \frac{6.3662}{r^2}\cos^3\theta = \frac{6.3662}{(10^3)^2}\cdot\cos^3\theta = 6.3662 \times 10^{-6}\cos^3\theta$

$W|_{max} = 6.3662 \times 10^{-6}\cdot\cos^3\theta\Big|_{max} = 6.3662 \times 10^{-6}$ watts/m²

(b) $D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi (6.3662)}{10} = 8 = 9\,dB$

(c) $G_0 = e_t D_0 = 8 = 9\,dB$

2-5

$$U(\theta,\phi) = \begin{cases} 1 & 0^\circ \le \theta \le 20^\circ \\ 0.342\,\csc(\theta) & 20^\circ \le \theta \le 60^\circ \\ 0 & 60^\circ \le \theta \le 180^\circ \end{cases} \quad 0^\circ \le \phi \le 360^\circ$$

$P_{rad} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta \, d\theta \, d\phi = 2\pi \left[ \int_0^{20^\circ} \sin\theta \, d\theta + \int_{20^\circ}^{60^\circ} 0.342\,\csc(\theta) \times \sin\theta \, d\theta \right] = 2\pi \left\{ -\cos\theta\Big|_0^{\pi/9} + 0.342\cdot\theta\Big|_{\pi/9}^{\pi/3} \right\}$

$= 2\pi \left\{ \left[-\cos\left(\frac{\pi}{9}\right)+1\right] + 0.342\left(\frac{\pi}{3}-\frac{\pi}{9}\right) \right\}$

$= 2\pi \left\{ \left[-0.93969+1\right] + 0.342\,\pi\left(\frac{2}{9}\right) \right\}$

$= 2\pi \left\{ 0.06031 + 0.23876 \right\} = 1.87912$

$D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi (1)}{1.87912} = 6.68737 = 8.25255\,dB.$

2-6 (a) $D_o \simeq \dfrac{41,253}{\theta_{1d}\,\theta_{2d}} = \dfrac{41,253}{30\,(35)} = 39.29 = 15.94\ dB$

$A_{em} = \dfrac{\lambda^2}{4\pi} D_o$

(b) $D_o \simeq \dfrac{72,815}{\theta_{1d}^2 + \theta_{2d}^2} = \dfrac{72,815}{(30)^2 + (35)^2} = 34.27 = 15.35\ dB$

$A_{em} = \dfrac{\lambda^2}{4\pi} D_o$

2-7 $D_o = \dfrac{4\pi\, U_{max}}{P_{rad}}$

(a) $U = \sin\theta \sin\phi$ for $0 \le \theta \le \pi$, $0 \le \phi \le \pi$

$U|_{max} = 1$ and it occurs when $\theta = \phi = \pi/2$.

$$P_{rad} = \int_0^{\pi}\int_0^{\pi} U \sin\theta\, d\theta d\phi = \int_0^{\pi} \sin\phi\, d\phi \int_0^{\pi} \sin^2\theta\, d\theta = 2\left(\frac{\pi}{2}\right) = \pi.$$

Thus $D_o = \dfrac{4\pi(1)}{\pi} = 4 = 6.02\ dB$

The half-power beamwidths are equal to

$HPBW(az.) = 2[90° - \sin^{-1}(1/2)] = 2(90° - 30°) = 120°$

$HPBW(el.) = 2[90° - \sin^{-1}(1/2)] = 2(90° - 30°) = 120°$

In a similar manner, it can be shown that for

(b) $U = \sin\theta \sin^2\phi \Rightarrow D_o = 5.09 = 7.07\ dB$

$HPBW(el.) = 120°$, $HPBW(az.) = 90°$

(c) $U = \sin\theta \sin^3\phi \Rightarrow D_o = 6 = 7.78\ dB$

$HPBW(el.) = 120°$, $HPBW(az.) = 74.93°$

(d) $U = \sin^2\theta \sin\phi \Rightarrow D_o = 12\pi/8 = 4.71 = 6.73\ dB$

$HPBW(el.) = 90°$, $HPBW(az.) = 120°$

(e) $U = \sin^2\theta \sin^2\phi \Rightarrow D_o = 6 = 7.78\ dB$, $HPBW(az.) = HPBW(el.) = 90°$

(f) $U = \sin^2\theta \sin^3\phi \Rightarrow D_o = 9\pi/4 = 7.07 = 8.49\ dB$

$HPBW(el.) = 90°$, $HPBW(az.) = 74.93°$

**2-8.** Using the half-power beamwidths found in the previous problem (problem 2-7), the directivity for each intensity using Kraus' and Tai and Pereira's formulas is given by

$U = \sin\theta \cdot \sin\phi$ ; (a) $D_o \simeq \dfrac{41\,253}{\Theta_{1d}\,\Theta_{2d}} = \dfrac{41\,253}{120\,(120)} = 2.86 = 4.57\ dB$

(b) $D_o \simeq \dfrac{72{,}815}{\Theta_{1d}^2 + \Theta_{2d}^2} = \dfrac{72{,}815}{(120)^2 + (120)^2} = 2.53 = 4.03\ dB$

$U = \sin\theta \cdot \sin^2\phi$ ; (a) $D_o \simeq 3.82 = 5.82\ dB$

(b) $D_o \simeq 3.24 = 5.10\ dB$

$U = \sin\theta \sin^3\phi$ ; (a) $D_o \simeq 4.59 = 6.62\ dB$

(b) $D_o \simeq 3.64 = 5.61\ dB$

$U = \sin^2\theta \cdot \sin\phi$ ; (a) $D_o \simeq 3.82 = 5.82\ dB$

(b) $D_o \simeq 3.24 = 5.10\ dB$

$U = \sin^2\theta \cdot \sin^2\phi$ ; (a) $D_o \simeq 5.09 = 7.07\ dB$

(b) $D_o \simeq 4.49 = 6.53\ dB$

$U = \sin^2\theta \cdot \sin^3\phi$ ; (a) $D_o \simeq 6.12 = 7.87\ dB$

(b) $D_o \simeq 5.31 = 7.25\ dB$

**2-9.**

(a) $D_o = \dfrac{4\pi}{\Theta_{1r}\,\Theta_{2r}} = \dfrac{4\pi}{(1.5064)^2} = 5.5377 = 7.433\ dB$

(b) $D_o = \dfrac{32 \cdot \ln(2)}{\Theta_{1r}^2 + \Theta_{2r}^2} = \dfrac{32 \cdot \ln(2)}{(1.5064)^2 + (1.5064)^2} = 4.88725$

$= 6.8906\ dB$

**2-10.** (a) $D_o = \dfrac{4\pi\, U_{max}}{P_{rad}} = \dfrac{U_{max}}{U_o}$

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi} U \sin\theta\, d\theta\, d\phi = 2\pi \int_0^{\pi} U \sin\theta\, d\theta = 2\pi \left\{ \int_0^{30^\circ} \sin\theta\, d\theta + \right.$$

$$\left. \int_{30^\circ}^{60^\circ} (0.5)\sin\theta\, d\theta + \int_{60^\circ}^{90^\circ} (0.1)\sin\theta\, d\theta \right\} = 2\pi \left\{ (-\cos\theta)\Big|_0^{30^\circ} + \left(-\frac{\cos\theta}{2}\right)\Big|_{30^\circ}^{60^\circ} + (-0.1\cos\theta)\Big|_{60^\circ}^{90^\circ} \right\}$$

(Continued)

2-10 (cont'd) $= 2\pi\left\{(-0.866+1)+\left(\frac{-0.5+0.866}{2}\right)+\left(\frac{-0+0.5}{10}\right)\right\}$

$P_{rad} = 2\pi\{-0.866+1-0.25+0.433+0.05\} = 2\pi(0.367)$

$= 0.734\cdot\pi = 2.3059$

$D_o = \frac{1(4\pi)}{2.3059} = 5.4496 = 7.3636\ dB$

(b) $D_o$ (dipole) $= 1.5 = 1.761\ dB$

$D_o$ (above dipole) $= (7.3636 - 1.761)\ dB = 5.6026\ dB$

$D_o$ (above dipole) $= \frac{5.45}{1.5} = 3.633 = 5.603\ dB$

2-11. (a) $P_{rad} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\sin^2\phi\, d\phi \cdot \int_0^{\pi/2}\cos^4\theta\sin\theta\, d\theta$

$= (\pi)\left(\frac{1}{5}\right) = \frac{\pi}{5}.$

$U_{max} = U(\theta=0°, \phi=\pi/2) = 1.$

$D_o = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi}{(\pi/5)} = 20 = 13.0\ dB$

(b) Elevation Plane: $\theta$ varies, $\phi$ fixed

→ choose $\phi = \pi/2$.

$U(\theta, \phi=\pi/2) = \cos^4\theta, \quad 0 \leq \theta \leq \pi/2.$

$\cos^4\left[\frac{HPBW(el.)}{2}\right] = \frac{1}{2}$

$HPBW(el.) = 2\cdot\cos^{-1}\{\sqrt{0.5}\} = 65.5°.$

2-12. (a) $P_{rad} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = 2\pi\cdot\left\{\int_0^{30°}\sin\theta\, d\theta + \int_{30°}^{90°}\frac{\cos\theta\cdot\sin\theta}{0.866}d\theta\right\}$

$= 2\pi\left\{\int_0^{\pi/6}\sin\theta\, d\theta + \int_{\pi/6}^{\pi/2}\frac{1}{0.866}\cos\theta\cdot\sin\theta\, d\theta\right\}$

$= 2\pi\left\{-\cos\theta\Big|_0^{\pi/6} + \frac{1}{0.866}\left(-\frac{\cos^2\theta}{2}\right)\Big|_{\pi/6}^{\pi/2}\right\} = 2\pi[-0.866+1+0.433] = 3.5626$

$D_o = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi(1)}{3.5626} = 3.5273 = 5.4745\ dB$

(b) $M = \frac{\cos(\theta)}{0.866} = 0.5 \Rightarrow \cos\theta = 0.5(0.866) = 0.433, \quad \theta = \cos^{-1}(0.433) = 64.34°$

$\Theta_{1r} = 2(64.34) = 128.68° = 2.246\ rad = \Theta_{2r}$

$D_o \simeq \frac{4\pi}{\Theta_{1r}\,\Theta_{2r}} = \frac{4\pi}{(2.246)^2} = 2.4912 = 3.9641\ dB$

2-13 a. 35dB

b. $20\log_{10}\left|\frac{E_{max}}{E_s}\right| = 35,\ \log_{10}\left|\frac{E_{max}}{E_s}\right| = \frac{35}{20} = 1.75$

$\left|\frac{E_{max}}{E_s}\right| = 10^{1.75} = 56.234$

2-14. a. $u=\sin\theta$, $U_{max}=1$, $P_{rad} = \int_0^{2\pi}\int_0^{\pi} u\sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi}\sin^2\theta\, d\theta\, d\phi = \pi^2$

$D_o = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi}{\pi^2} = \frac{4}{\pi} = 1.2732$

b. HPBW = 120°, $2\pi/3$

The directivity based on (2-33a) is equal to, $D_o = \frac{101}{120° - 0.0027(120°)^2} = 1.2451$

While that based on (2-33b) is equal to, $D_o = -172.4 + 191\sqrt{0.818 + \frac{1}{120°}} = 1.2245$

c. Computer Program $D_o = 1.2732$

2-15. a. $u = \sin^3\theta$, $U_{max}=1$, $P_{rad} = \int_0^{2\pi}\int_0^{\pi}\sin^4\theta\, d\theta\, d\phi = \frac{3}{4}\pi^2$, $D_o = \frac{4\pi}{\frac{3}{4}\pi^2} = \frac{16}{3\pi} = 1.6976$

b. HPBW = 74.93°

From (2-33a), $D_o = \frac{101}{(74.93°) - 0.0027(74.93°)^2} = 1.68971$

From (2-33b), $D_o = -172.4 + 191\sqrt{0.818 + \frac{1}{74.93°}} = 1.75029$

c. Computer program $D_o = 1.69766$

The value of $D_o$ (1.6976) is similar to that of (4-91) or 1.643

2-16. a. $u = J_1^2(ka\sin\theta)$,

$a = \lambda/10$, $ka\sin\theta = \frac{\pi}{5}\sin\theta$. HPBW = 93.10°

From (2-33a) $D_o = 101/[(93.10) - 0.0027(93.10)^2] = 1.449120$

From (2-33b) $D_o = -172.4 + 191\sqrt{0.818 + \frac{1}{93.10}} = 1.477271$

$a = \lambda/20$, $ka\sin\theta = \frac{\pi}{10}\sin\theta$, HPBW = 91.10°.

From (2-33a), $D_o = 1.47033$, From (2-33b), $D_o = 1.502$

b. $a = \frac{\lambda}{10}$. $P_{rad} = \int_0^{2\pi}\int_0^{\pi} J_1^2(ka\sin\theta)\cdot\sin\theta\, d\theta\, d\phi = 0.7638045$

$U_{max} = 0.0893$, $\therefore D_o = \frac{4\pi(0.0893)}{0.7638045} = 1.469193$

$a = \frac{\lambda}{20}$, $P_{rad} = \int_0^{2\pi}\int_0^{\pi} J_1^2(\pi/10\cdot\sin\theta)\cdot\sin\theta\, d\theta\, d\phi = 0.202604$

$U_{max} = 0.0240714$, $D_o = \frac{4\pi(0.0240714)}{0.202604} = 1.49257$.

If the radius of loop is smaller than $\lambda/20$, the directivity approach to 1.5.

2-17. Using the numerical techniques, the directivity for each intensity of (Prob. 2-7) with $10^\circ$ uniform divisions is equal to

$U = \sin\theta \cdot \sin\phi$ ; (a) Midpoint ; $D_o = \dfrac{4\pi U_{max}}{P_{rad}}$

$U_{max} = 1.$ $\quad P_{rad} = \dfrac{\pi}{18}\left(\dfrac{\pi}{18}\right)\sum_{j=1}^{18} \sin\phi_j \sum_{i=1}^{18} \sin^2\theta_i$

$\theta_i = \dfrac{\pi}{36} + (i-1)\dfrac{\pi}{18}, \quad i = 1, 2, 3, \cdots, 18$

$\phi_j = \dfrac{\pi}{36} + (j-1)\dfrac{\pi}{18}, \quad j = 1, 2, 3, \cdots, 18$

$P_{rad} = \left(\dfrac{\pi}{18}\right)^2 (11.38656)(8.9924) = 3.119$

$D_o = \dfrac{4\pi(1)}{3.119} = 4.03 = 6.05\ dB$

(b) Trailing edge of each division

Trailing edge: $\theta_i = i(\pi/18), \quad i = 1, 2, 3, \cdots, 18$

$\phi_j = j(\pi/18), \quad j = 1, 2, 3, \cdots, 18$

$P_{rad} = \left(\dfrac{\pi}{18}\right)^2 (11.25640)(8.96985) = 3.076$

$D_o = \dfrac{4\pi(1)}{3.119} = 4.09 = 6.11\ dB$

In a similar manner

$U = \sin\theta \sin^2\phi$ ; (a) $P_{rad} = 2.463 \Rightarrow D_o = 5.10 = 7.07\ dB$

(b) $P_{rad} = 2.451 \Rightarrow D_o = 5.13 = 7.10\ dB$

$U = \sin\theta \sin^3\phi$ ; (a) $P_{rad} = 2.092 \Rightarrow D_o = 6.01 = 7.79\ dB$

(b) $P_{rad} = 2.086 \Rightarrow D_o = 6.02 = 7.80\ dB$

$U = \sin^2\theta \cdot \sin\phi$ ; (a) $P_{rad} = 2.469 \Rightarrow D_o = 4.74 = 6.76\ dB$

(b) $P_{rad} = 2.618 \Rightarrow D_o = 4.80 = 6.81\ dB$

$U = \sin^2\theta \cdot \sin^2\phi$ ; (a) $P_{rad} = 2.092 \Rightarrow D_o = 6.01 = 7.79\ dB$

(b) $P_{rad} = 2.086 \Rightarrow D_o = 6.02 = 7.80\ dB$

$U = \sin^2\theta \cdot \sin^3\phi$ ; (a) $P_{rad} = 1.777 \Rightarrow D_o = 7.07 = 8.49\ dB$

(b) $P_{rad} = 1.775 \Rightarrow D_o = 7.08 = 8.50\ dB$

2-18. Using the computer program Directivity at the end of chapter 2, the directivities for each radiation intensity of Problem 2.7 are equal to

a. $U = \sin\theta \sin\phi$ ; $P_{rad} = 3.1318$

$U_{max} = 1$ ; $D_o = \frac{4\pi \cdot U_{max}}{3.1318} = 4.0125 \Rightarrow 6.034\,dB$

b. $U = \sin\theta \cdot \sin^2\phi$ ; $P_{rad} = 2.4590$

$U_{max} = 1$ ; $D_o = \frac{4\pi \cdot 1}{2.4590} = 5.110358 \Rightarrow 7.0845\,dB$

c. $U = \sin\theta \cdot \sin^3\phi$ ; $P_{rad} = 2.0870$

$U_{max} = 1$ ; $D_o = \frac{4\pi \cdot 1}{2.0870} = 6.02124 \Rightarrow 7.80\,dB$

d. $U = \sin^2\theta \sin\phi$ ; $P_{rad} = 2.6579$

$U_{max} = 1$ ; $D_o = \frac{4\pi \cdot 1}{2.6579} = 4.72793 \Rightarrow 6.746\,dB$

e. $U = \sin^2\theta \sin^2\phi$ ; $P_{rad} = 2.0870$

$D_o = \frac{4\pi \cdot 1}{2.0870} = 6.02126 \Rightarrow 7.7968\,dB$

f. $U = \sin^2\theta \cdot \sin^3\phi$ ; $P_{rad} = 1.7714$

$D_o = \frac{4\pi \cdot 1}{1.7714} = 7.09403 \Rightarrow 8.5089\,dB$

2-19. (a). $E|_{max} = \cos\left[\frac{\pi}{4}(\cos\theta - 1)\right]\Big|_{max} = 1$ at $\theta = 0°$.

$0.707\, E_{max} = 0.707 \cdot (1) = \cos\left[\frac{\pi}{4}(\cos\theta_1 - 1)\right]$

$\frac{\pi}{4}(\cos\theta_1 - 1) = \pm\frac{\pi}{4} \Rightarrow \theta_1 = \begin{cases} \cos^{-1}(2) = \text{does not exist} \\ \cos^{-1}(0) = 90° = \frac{\pi}{2} \text{ rad.} \end{cases}$

$\Theta_{1r} = \Theta_{2r} = 2\left(\frac{\pi}{2}\right) = \pi$

$D_o \simeq \frac{4\pi}{\Theta_{1r}\Theta_{2r}} = \frac{4\pi}{\pi^2} = \frac{4}{\pi} = 1.273 = 1.049 \text{ dB}$

(b). Using the computer program at the end of chapter 2

$D_o = 2.00789 = 3.027$ dB

Since the pattern is not very narrow, the answer obtained using Kraus' approximate formula is not as accurate.

2-20. a. $E|_{max} = \cos\left(\frac{\pi}{4}(\cos\theta + 1)\right)\Big|_{max} = 1$ at $\theta = \pi$.

$0.707 \cdot = \cos\left(\frac{\pi}{4}(\cos\theta_1 + 1)\right)$

$\frac{\pi}{4}(\cos\theta_1 + 1) = \pm\frac{\pi}{4} \Rightarrow \theta_1 = \begin{cases} \cos^{-1}(-2) \rightarrow \text{does not exist.} \\ \cos^{-1}(0) \rightarrow 90° \rightarrow \frac{\pi}{2} \text{ rad} \end{cases}$

$\Theta_{1r} = \Theta_{2r} = 2\left(\frac{\pi}{2}\right) = \pi.$

$D_o \simeq \frac{4\pi}{\pi^2} = \frac{4}{\pi} = 1.273 = 1.048 \text{ dB}$

b. Computer Program

$D_o = 2.00789 = 3.027$ dB

2-21. a. $P_{rad} = \int_0^{2\pi}\int_0^{\pi/2} U_o \sin(\pi\sin\theta)\cdot\sin\theta\, d\theta\, d\phi = 2\pi \cdot U_o \cdot \frac{\pi}{2} J_1(\pi) = U_o \pi^2 J_1(\pi)$

$D_o = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi U_o}{U_o \pi^2 J_1(\pi)} = \frac{4}{\pi}\cdot\frac{1}{J_1(\pi)} = 4.4735$

$\leftarrow \frac{\pi}{2} J_1(\pi) = 0.44707273561622$

b. Computer Program

$P_{rad} = \int_0^{2\pi}\int_0^{\pi/2} U_o \sin(\pi\sin\theta)\sin\theta\, d\theta d\phi = 2\pi\cdot(0.44707273561618)$

$D_o = 4.4735$

**2-22.** (a). Using the computer program at the end of Chapter 2.

$$D_o = 14.0707 = 11.48 \text{ dB}$$

(b). $U|_{max} = \left[\frac{\sin(\pi \sin\theta)}{\pi \sin\theta}\right]^2_{max} = 1$ when $\theta = 0°$.

$$U = \frac{1}{2} U_{max} = \frac{1}{2}(1) = \left[\frac{\sin(\pi \sin\theta_1)}{\pi \sin\theta_1}\right]^2$$

Iteratively we obtain $\theta_1 = 26.3°$, Therefore

$$\text{HP}_{1d} = \text{HP}_{2d} = 2(26.3°) = 52.6°.$$

and $D_o \simeq \frac{41,253}{(52.6)^2} = 14.91 = 11.73$ dB using the Kraus' formula

(c) For Tai and Pereira's formula

$$D_o \simeq \frac{72,815}{2 \cdot \text{HP}^2_{1d}} = \frac{72,815}{2(52.6)^2} = 13.16 = 11.19 \text{ dB}$$

**2-23** $U = \frac{1}{2\eta}|E|^2 = \frac{1}{2\eta}\sin\theta\cos^2\phi \Rightarrow U_{max} = \frac{1}{2\eta}$

(a). $P_{rad} = 2\cdot\int_0^{\pi/2}\int_0^{\pi} \frac{1}{2\eta}\sin^2\theta\cos^2\phi \, d\theta \, d\phi = \frac{1}{\eta}\left(\frac{\pi}{4}\right)\left(\frac{\pi}{2}\right) = \frac{\pi^2}{8\eta}$

$$D_o = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi\left(\frac{1}{2\eta}\right)}{\frac{\pi^2}{8\eta}} = \frac{16}{\pi} = 5.09 = 7.07 \text{ dB}$$

(b). $U_{max} = \frac{1}{2\eta}$ at $\theta = \pi/2$, $\phi = 0$

In the elevation plane through the maximum $\phi = 0$ and $U = \frac{1}{2\eta}\sin\theta$.

The 3-dB point occurs when

$U = 0.5\, U_{max} = 0.5\left(\frac{1}{2\eta}\right) = \frac{1}{2\eta}\sin\theta_1 \Rightarrow \theta_1 = \sin^{-1}(0.5) = 30°$

Therefore $\text{HP}_{1d} = 2(90-30) = 120°$

In the azimuth plane through the maximum $\theta = \pi/2$ and $U = \frac{1}{2\eta}\cos^2\phi$.

The 3-dB point occurs when $U = 0.5\, U_{max} = 0.5\left(\frac{1}{2\eta}\right) = \frac{1}{2\eta}\cos^2\theta_1 \Rightarrow$

$\phi_1 = \cos^{-1}(0.707) = 45°$, $\text{HP}_{2d} = 2(90°-45°) = 90°$.

Therefore using Kraus' formula $D_o \simeq \frac{41,253}{120\cdot(90)} = 3.82 = 5.82$ dB

(c) Using Tai and Pereira's formula

$$D_o \simeq \frac{72,815}{\text{HP}^2_{1d} + \text{HP}^2_{2d}} = \frac{72,815}{(120)^2+(90)^2} = 3.24 = 5.10 \text{ dB}$$

(d) Using the computer program at the end of Chapter 2.

$$D_o = 5.16425 = 7.13 \text{ dB}$$

2-24. $U = \left[\frac{J_1(ka\sin\theta)}{\sin\theta}\right]^2 = (ka)^2\left[\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right]^2 = U_o\left[\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right]^2$

(a) $U_{max} = U_o(\frac{1}{2})^2 = \frac{U_o}{4}$ and it occurs when $ka\sin\theta = 0 \Rightarrow \theta = 0°$.

The 3-dB point is obtained using

$$U = \frac{1}{2}U_{max} = \frac{U_o}{8} = U_o\left[\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right]^2 \Rightarrow \frac{J_1(ka\sin\theta)}{ka\sin\theta} = 0.3535$$

With the aid of the $J_1(x)/x$ tables of Appendix V

$$X = ka\sin\theta_1 = 1.61 \Rightarrow \theta_1 = \sin^{-1}(1.61/2\pi) = 14.847° \Rightarrow \Theta_{1r} = 29.694°$$

(b) Since $\Theta_{1r} = \Theta_{2r} = 29.694°$, the directivity is equal to

$$D_o \simeq \frac{41,253}{(29.694)^2} = 46.79 = 16.70 \text{ dB}$$

2-25. (a) Linear because $\Delta\phi = 0$.

(b) Linear because $\Delta\phi = 0$.

(c) Circular because 1. $E_x = E_y$

2. $\Delta\phi = \pi/2$.

CCW because $E_y$ leads $E_x$, $AR = 1$, $\tau = 90°$

(d) Circular because 1. $E_x = E_y$

2. $\Delta\phi = -\pi/2$

CW because $E_y$ lags $E_x$. $AR = 1$, $\tau = 90°$

(e) Elliptical because $\Delta\phi$ is not multiples of $\pi/2$.

CCW because $E_y$ leads $E_x$. $AR = OA/OB$

Letting $E_x = E_y = E_o$

$$\left.\begin{aligned} OA &= E_o[0.5(1+1+\sqrt{2})]^{1/2} = 1.30656\,E_o \\ OB &= E_o[0.5(1+1-\sqrt{2})]^{1/2} = 0.541196\,E_o \end{aligned}\right\} \Rightarrow AR = \frac{1.30656}{0.541196} = 2.414$$

$$\tau = 90° - \frac{1}{2}\tan^{-1}\left[\frac{2(1)\cos(45°)}{1-1}\right] = 90° - \frac{1}{2}\tan^{-1}\left(\frac{1.414}{0}\right)$$

$$= 90° - \frac{1}{2}(90°) = 45°$$

(f) Elliptical because $\Delta\phi$ is not multiples of $\pi/2$

CW because $E_y$ lags $E_x$

From above $\left.\begin{aligned} OA &= 1.30656\,E_o \\ OB &= 0.541196\,E_o \end{aligned}\right\} \Rightarrow AR = \frac{1.30656}{0.541196} = 2.414$

From above $\tau = 90° - \frac{1}{2}(90°) = 45°$

(Continued)

2-25 (cont'd) (g). Elliptical because 1. $E_x \neq E_y$

2. $\Delta\phi$ is not zero or multiples of $\pi$.

CCW because $E_y$ leads $E_x$.

$$OA = E_y\left\{\tfrac{1}{2}[0.25+1+0.75]\right\}^{1/2} = E_y$$
$$OB = E_y\left\{\tfrac{1}{2}[0.25+1-0.75]\right\}^{1/2} = 0.5\,E_y$$
$$\Rightarrow AR = \frac{1}{0.5} = 2.$$

$$\tau = 90^\circ - \tfrac{1}{2}\tan^{-1}\left(\frac{0}{-0.75}\right) = 90^\circ - \tfrac{1}{2}(180^\circ) = 0^\circ$$

(h) Elliptical because 1. $E_x \neq E_y$

2. $\Delta\phi$ is not zero or multiples of $\pi$.

CW because $E_y$ lags $E_x$.

From above $OA = E_y$, $OB = 0.5\,E_y$ $\Rightarrow AR = \frac{1}{0.5} = 2$

$$\tau = 90^\circ - \tfrac{1}{2}(180^\circ) = 0^\circ.$$

2-26. $$E_x(z,t) = \mathrm{Re}\left[E_x e^{j(\omega t + kz + \phi_x)}\right] = E_x \cos(\omega t + kz + \phi_x)$$
$$E_y(z,t) = \mathrm{Re}\left[E_y e^{j(\omega t + kz + \phi_y)}\right] = E_y \cos(\omega t + kz + \phi_y)$$

where $E_x$ and $E_y$ are real positive constants.

Choosing $z=0$ and letting $\Delta\phi = \phi_y - \phi_x = \phi_y - 0 = \phi$

$$\mathcal{E}_x(t) = E_x \cos(\omega t)$$
$$\mathcal{E}_y(t) = E_y \cos(\omega t + \phi) \qquad (1)$$

and

$$\mathcal{E}(t) = \sqrt{\mathcal{E}_x^2 + \mathcal{E}_y^2} = \sqrt{E_x^2\cos^2(\omega t) + E_y^2\cos^2(\omega t+\phi)} \qquad (2)$$

The maximum and minimum values of (2) are the major and minor axes of the polarization ellipse. Squaring (2) and using the half-angle identity, equation (2) can be written as

$$\mathcal{E}^2(t) = \tfrac{1}{2}\left\{E_x^2 + E_y^2 + E_x^2\cos(2\omega t) + E_y^2\cos^2[2(\omega t+\phi)]\right\} \qquad (3)$$

Since $E_x$ and $E_y$ are constants, the maximum and minimum values of (3) occur when $f(t) = E_x^2\cos(2\omega t) + E_y^2\cos[2(\omega t+\phi)]$ (4) is maximum or minimum. These are found by differentiating (4) and setting it equal to zero. Thus

$$\frac{df}{d(2\omega t)} = -E_x^2\sin(2\omega t) - E_y^2\sin[2(\omega t+\phi)] = 0$$

(Continued)

2-26 (Cont'd) or

$$E_x^2 \sin(2\omega t) = -E_y^2 \sin[2(\omega t+\phi)]$$
$$= -E_y^2\{\sin 2\omega t \cos 2\phi + \cos 2\omega t \sin 2\phi\} \quad (5)$$

Dividing (5) by $\cos(2\omega t)$ yields

$$E_x^2 \tan(2\omega t) = -E_y^2 \tan[2\omega t]\cos(2\phi) + \sin(2\phi)]$$

or

$$\tan(2\omega t) = \frac{-E_y^2 \sin(2\phi)}{E_x^2 + E_y^2 \cos(2\phi)}$$

from which we obtain that

$$\cos(2\omega t) = \frac{E_x^2 + E_y^2 \cos(2\phi)}{\pm\rho} \quad (6)$$

$$\cos(2\omega t + 2\phi) = \frac{E_y^2 + E_x^2 \cos(2\phi)}{\pm\rho} \quad (7)$$

where $\rho = \sqrt{E_x^4 + E_y^4 + 2E_x^2 E_y^2 \cos(2\phi)}$ (8)

Substituting (6)-(8) into (3) yields

$$\mathcal{E}^2 = \frac{1}{2}\left[E_x^2 + E_y^2 \pm \frac{1}{\rho}(\rho^2)\right]$$

whose maximum value is

$$\mathcal{E}_{max} = OA = \left\{\frac{1}{2}\left[E_x^2 + E_y^2 + (E_x^4 + E_y^4 + 2E_x^2E_y^2\cos 2\phi)^{1/2}\right]\right\}^{1/2}$$

$$\mathcal{E}_{min} = OB = \left\{\frac{1}{2}\left[E_x^2 + E_y^2 - (E_x^4 + E_y^4 + 2E_x^2E_y^2\cos 2\phi)^{1/2}\right]\right\}^{1/2}$$

The tilt angle T can be obtained by expanding (1) and writing the two as

$$\frac{E_x^2}{E_x^2} - \frac{2E_x E_y \cos\phi}{E_x E_y} + \frac{\mathcal{E}_y^2}{E_y^2} = \sin^2\phi \quad (9)$$

which is the equation of a tilted ellipse. Choosing a coordinate system whose principal axes coincide with the major and minor axes of the tilted ellipse, we can write that

$$\mathcal{E}_x = \mathcal{E}_{x'}\sin(\tau) - \mathcal{E}_{y'}\cos(\tau)$$
$$\mathcal{E}_y = \mathcal{E}_{x'}\cos(\tau) + \mathcal{E}_{y'}\sin(\tau) \quad (10)$$

where $\mathcal{E}_{x'}$ and $\mathcal{E}_{y'}$ are the new field values along the new principal axes $x', y', z'$. Substituting (10) into (9) yields

$$\frac{2E_{x'}E_{y'}\cos(\tau)\sin(\tau)}{E_x^2} - \frac{2E_{x'}E_{y'}\cos(\tau)\sin(\tau)}{E_y^2} - \frac{2E_{x'}E_{y'}\cos\phi}{E_x E_y}(\sin^2\tau - \cos^2\tau) = 0$$

(Continued)

2-26. (cont'd)

which when solved for the tilt angle $\tau$ reduces to

$$\tan[2(\tfrac{\pi}{2}-\tau)] = \frac{2E_x E_y \cos\phi}{E_x^2 - E_y^2}$$

or $$\tau = \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left(\frac{2E_x E_y \cos\phi}{E_x^2 - E_y^2}\right)$$

For more details on the tilt angle derivation, see

J. D. Kraus, Antennas, McGraw-Hill, 1950, pp. 464-476.

2-27. (a) $\hat{\rho}_w = \hat{a}_x \cos\phi_1 + \hat{a}_y \sin\phi_1$

$\hat{\rho}_a = \hat{a}_x \cos\phi_2 + \hat{a}_y \sin\phi_2$

$$PLF = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = |(\hat{a}_x \cos\phi_1 + \hat{a}_y \sin\phi_1)\cdot(\hat{a}_x \cos\phi_2 + \hat{a}_y \sin\phi_2)|^2$$
$$= |\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2|^2 = |\cos(\phi_1 - \phi_2)|^2$$

(b) $\hat{\rho}_w = \hat{a}_x \sin\theta_1 \cos\phi_1 + \hat{a}_y \sin\theta_1 \sin\phi_1 + \hat{a}_z \cos\theta_1$

$\hat{\rho}_a = \hat{a}_x \sin\theta_2 \cos\phi_2 + \hat{a}_y \sin\theta_2 \sin\phi_2 + \hat{a}_z \cos\theta_2$

$$PLF = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = |\sin\theta_1 \cos\phi_1 \sin\theta_2 \cos\phi_2 + \sin\theta_1 \sin\phi_1 \sin\theta_2 \cdot \sin\phi_2 + \cos\theta_1 \cdot \cos\theta_2|^2$$

$$PLF = |\sin\theta_1 \sin\theta_2 (\cos\phi_1 \cdot \cos\phi_2 + \sin\phi_1 \sin\phi_2) + \cos\theta_1 \cos\theta_2|^2$$

$$PLF = |\sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2) + \cos\theta_1 \cos\theta_2|^2$$

2-28. Assuming electric field is x-polarized

(a) $\underline{E}_w = \hat{a}_x E_1 e^{-jkz} \Rightarrow \hat{\rho}_w = \hat{a}_x$

$\underline{E}_a \simeq (\hat{a}_\theta - j\hat{a}_\phi) E_0 f(r,\theta,\phi) \Rightarrow \hat{\rho}_a = \left(\frac{\hat{a}_\theta - j\hat{a}_\phi}{\sqrt{2}}\right)$

$$PLF = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \tfrac{1}{2}|\hat{a}_x \cdot \hat{a}_\theta - j\hat{a}_x \cdot \hat{a}_\phi|^2$$

since $\hat{a}_\theta = \hat{a}_x \cos\theta\cos\phi + \hat{a}_y \cos\theta \sin\phi - \hat{a}_z \sin\theta$

$\hat{a}_\phi = -\hat{a}_x \sin\phi + \hat{a}_y \cos\phi$

$$PLF = \tfrac{1}{2}(\cos^2\theta \cos^2\phi + \sin^2\phi)$$

(b) when $\underline{E}_a \simeq (\hat{a}_\theta + j\hat{a}_\phi) E_0 f(r,\theta,\phi)$, PLF is also

$$PLF = \tfrac{1}{2}(\cos^2\theta \cos^2\phi + \sin^2\phi)$$

A more general, but also more complex, expression can be derived when the incident electric field is of the form $\underline{E}_w = (a\hat{a}_x + b\hat{a}_y)e^{-jkz}$ where $a, b$ are real constants. It can be shown (using the same procedure) that

$$PLF = \frac{1}{\sqrt{2(a^2+b^2)}}\left[(a\cos\theta\cos\phi + b\sin\theta\sin\phi)^2 + (a\sin\phi - b\cos\phi)^2\right]^{1/2}$$

2-29. (a) $\underline{E}_w = E_0(\hat{a}_x \pm j\hat{a}_y)e^{-jkz} \Rightarrow \hat{\rho}_w = \frac{1}{\sqrt{2}}(\hat{a}_x \pm j\hat{a}_y)$

$\underline{E}_a \simeq E_1(\hat{a}_\theta - j\hat{a}_\phi) f(r,\theta,\phi) \Rightarrow \hat{\rho}_a = \frac{1}{\sqrt{2}}(\hat{a}_\theta - j\hat{a}_\phi)$

$PLF = \frac{1}{2}\left|(\hat{a}_x \pm j\hat{a}_y)\cdot(\hat{a}_\theta - j\hat{a}_\phi)\right|^2 = \frac{1}{2}\left|(\hat{a}_x\cdot\hat{a}_\theta \pm \hat{a}_y\cdot\hat{a}_\phi) - j(\hat{a}_x\cdot\hat{a}_\phi \mp \hat{a}_y\cdot\hat{a}_\theta)\right|^2$

Converting the spherical unit vectors to rectangular, as it was done in Problem 2.27, leads to

$PLF = \frac{1}{2}(\cos\theta \pm 1)^2$

(b) When

$\underline{E}_w = E_0(\hat{a}_x \pm j\hat{a}_y)e^{-jkz}$

$\underline{E}_a \simeq E_1(\hat{a}_\theta + j\hat{a}_\phi) f(r,\theta,\phi)$ the PLF is equal to

$PLF = \frac{1}{2}(\cos\theta \mp 1)^2$

2-30. $\underline{E}_w = (\hat{a}_\theta\cos\phi - \hat{a}_\phi\sin\phi\cos\theta) f(r,\theta,\phi)$

or $\underline{E}_w = \left[\frac{\hat{a}_\theta\cos\phi - \hat{a}_\phi\sin\phi\cos\theta}{\sqrt{\cos^2\phi + \sin^2\phi\cos^2\theta}}\right]\sqrt{\cos^2\phi + \sin^2\phi\cos^2\theta}\cdot f(r,\theta,\phi)$

Thus $\hat{\rho}_w = \frac{\hat{a}_\theta\cos\phi - \hat{a}_\phi\sin\phi\cos\theta}{\sqrt{\cos^2\phi + \sin^2\phi\cos^2\theta}}$

and $PLF = |\hat{\rho}_w\cdot\hat{\rho}_a|^2 = \left|\left(\frac{\hat{a}_\theta\cos\phi - \hat{a}_\phi\sin\phi\cos\theta}{\sqrt{\cos^2\phi + \sin^2\phi\cos^2\theta}}\right)\cdot\hat{a}_x\right|^2$

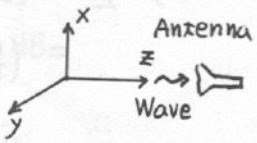
Transforming the rectangular unit vector to spherical using

$\hat{a}_x = \hat{a}_r\sin\theta\cos\phi + \hat{a}_\theta\cos\theta\cos\phi - \hat{a}_\phi\sin\phi$

the PLF reduces to $PLF = \frac{\cos^2\theta}{\cos^2\phi + \sin^2\phi\cos^2\theta}$

The same answer is obtained by transforming the spherical unit vectors to rectangular, as was done in Prob 2-27.

2-31. $\underline{E}_a \simeq (2\hat{a}_x \pm j\hat{a}_y) f(r,\theta,\phi) = \left(\frac{2\hat{a}_x \pm j\hat{a}_y}{\sqrt{5}}\right)\sqrt{5}\, f(r,\theta,\phi)$




(a) $\hat{\rho}_w = \left(\frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right) \Rightarrow$ Wave is Right Hand (RH)

$\hat{\rho}_a = \left(\frac{2\hat{a}_x \pm j\hat{a}_y}{\sqrt{5}}\right)$

(Continued) $PLF = |\hat{\rho}_w\cdot\hat{\rho}_a|^2 = \begin{cases} \frac{9}{10} = -0.4576\ dB & \text{using the + sign} \\ \frac{1}{10} = -10\ dB & \text{using the − sign} \end{cases}$

(Antenna is LH in receiving mode and RH in transmitting)

(Antenna is RH in receiving mode and LH in transmitting)

2-31. (cont'd) (b) $\hat{\rho}_w = \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right) \Rightarrow$ Wave is Left Hand (LH)

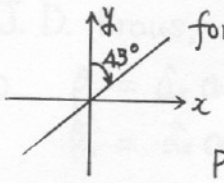
$$\hat{\rho}_a = \left(\frac{2\hat{a}_x \pm j\hat{a}_y}{\sqrt{5}}\right)$$

$$PLF = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \begin{cases} \frac{1}{10} = -10\,dB & \text{Using the + Sign} \\ \frac{9}{10} = -0.4545\,dB & \text{Using the − Sign} \end{cases}$$

(Antenna is LH in receiving mode and RH in transmitting)

(Antenna is RH in receiving mode and LH in transmitting)

2-32.




for $\hat{\rho}_w$

$$\hat{\rho}_w = \frac{\hat{a}_x + \hat{a}_y}{\sqrt{2}}; \quad PLF = \left|\frac{\hat{a}_x + \hat{a}_y}{\sqrt{2}} \cdot \frac{4\hat{a}_x + j\hat{a}_y}{\sqrt{17}}\right|^2$$

$$PLF = \frac{1}{34}\left|(\hat{a}_x \cdot 4\hat{a}_x) + (\hat{a}_y \cdot j\hat{a}_y)\right|^2 = \frac{1}{34}|4 + j|^2$$

$$= 0.5$$

2-33. (a) RHCP; $\hat{\rho}_a = \frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}$

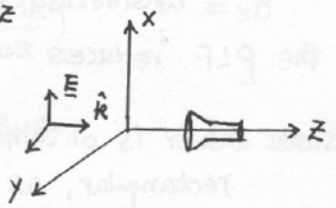
$$PLF = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\frac{2\hat{a}_x + j\hat{a}_y}{\sqrt{5}} \cdot \frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right|^2 = 0.9 = -0.46\ (dB)$$

(b) LHCP; $\hat{\rho}_a = \frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}$

$$PLF = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\frac{2\hat{a}_x + j\hat{a}_y}{\sqrt{5}} \cdot \frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right|^2 = 0.1 = -10.0\ (dB)$$

2-34. $\underline{E}^i = (\hat{a}_x - j\hat{a}_y) E_0 e^{-jkz} = \left(\frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right)\sqrt{2}\, E_0 e^{-jkz}$

$$\hat{\rho}_w = \frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}} \qquad CW$$




(a) $\underline{E}^a = (\hat{a}_x + j\hat{a}_y) E_1 e^{+jkz}$

$$= \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\sqrt{2}\, E_1 e^{+jkz}$$

$$\hat{\rho}_a = \frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}} \qquad CW$$

$$PLF = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\left(\frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right) \cdot \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\right|^2 = \left(\frac{1 - j^2}{2}\right)^2 = 1$$

$$PLF = 1 = 0\ dB$$

(Continued)

2-34. (Cont'd) (b) $\underline{E}^a = \left(\frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right)\sqrt{2}\,E_1 e^{+jkz}$

$$\hat{\rho}_a = \frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}$$

$$PLF = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\left(\frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right)\cdot\left(\frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right)\right|^2 = \left|\frac{1 + j^2}{2}\right|^2 = 0$$

$$PLF = 0 = -\infty\ dB$$

2-35. $\underline{E}^i = \hat{a}_x E_0 e^{-jkz}$ , $\hat{\rho}_w = \hat{a}_x$

$$\underline{E}^a = (\hat{a}_x + j\hat{a}_y) E_1 e^{+jkz} = \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\sqrt{2}\,E_1 e^{+jkz}$$

$$\hat{\rho}_a = \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)$$

(a) $A_{em} = \frac{\lambda^2}{4\pi} e_t D_0 |\hat{\rho}_a \cdot \hat{\rho}_w|^2 = \frac{\lambda^2}{4\pi} G_0 |\hat{\rho}_a \cdot \hat{\rho}_w|^2$ ($\leftarrow e_t D_0 = G_0$)

At 10 GHz $\Rightarrow \lambda = \frac{c}{f} = \frac{3\times10^8}{10\times10^9} = \frac{3\times10^8}{10^{10}} = 3\times10^{-2}$

$$G_0 = 10 = 10\log_{10} G_0(dim) \Rightarrow G_0(dim) = 10^1 = 10$$

$$A_{em} = \frac{\lambda^2}{4\pi} G_0 |\hat{\rho}_a \cdot \hat{\rho}_w|^2 = \frac{(3\times10^{-2})^2}{4\pi}(10)\left|\hat{a}_x \cdot \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\right|^2$$

$$= \frac{9\times10^{-4}}{4\pi}(10)\left(\frac{1}{2}\right) = \frac{9\times10^{-3}}{4\pi}\left(\frac{1}{2}\right) = (0.7162\times10^{-3})\left(\frac{1}{2}\right)$$

$$A_{em} = 0.3581\times10^{-3}\ m^2$$

(b) $P_T = A_{em}\, W^i = (0.3581\times10^{-3})(10\times10^{-3}) = 3.581\times10^{-6}$ watts

$$P_T = 3.581\times10^{-6}\ \text{watts}$$

2-36. $\underline{E}_a = (2\hat{a}_x \pm j\hat{a}_y) E e^{-jkz}$

$$\hat{\rho}_a = \frac{2\hat{a}_x \pm j\hat{a}_y}{\sqrt{5}}$$

(a) $\underline{E}_w = \hat{a}_x E_w \Rightarrow \hat{\rho}_w = \hat{a}_x$

$$PLF = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\frac{2}{\sqrt{5}}\right|^2 = \frac{4}{5} = 0.8 = -0.9691\ dB$$

(b) $\underline{E}_w = \hat{a}_y E_w \Rightarrow \hat{\rho}_w = \hat{a}_y$

$$PLF = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\frac{1}{\sqrt{5}}\right|^2 = \frac{1}{5} = 0.2 = -6.9897\ dB$$

2-37. (a) $E_y = E_y' + E_y'' = 3\cos\omega t + 2\cos\omega t = 5\cos\omega t$

$E_x = E_x' + E_x'' = 7\cos(\omega t + \frac{\pi}{2}) + 3\cos(\omega t - \frac{\pi}{2})$

$= -7\sin\omega t + 3\sin\omega t = -4\sin\omega t$

$AR = \frac{5}{4} = 1.25$

(b) At $\omega t = 0$, $\vec{E} = 5\hat{a}_y$

At $\omega t = \pi/2 \Rightarrow \vec{E} = -4\hat{a}_x \Rightarrow$ Rotation in CCW

2-38. (a) PLF $= \frac{1}{2}$ independent of $\psi \rightarrow$ must have CP

$\therefore AR = 1.$

(b). Polarization will be elliptical with major axies aligned with X-axis.

guess : $AR = 2$

verify : $\hat{\rho}_w = (2\hat{a}_x + j a_y)/\sqrt{5}$

$PLF = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\frac{2\cos\psi + j\sin\psi}{\sqrt{5}}\right|^2 = \frac{4\cos^2\psi + \sin^2\psi}{5}$

$\psi = 0$ : PLF $= 0.8$

$\psi = 90°$ : PLF $= 0.2$

(C) PLF $= 1$ at $\psi = 45°$ and $225°$

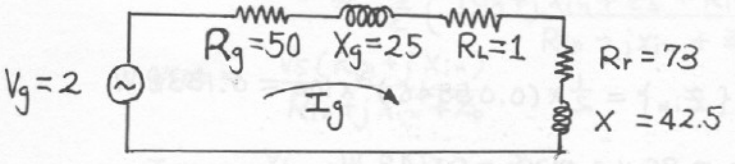
PLF $= 0$ at $\psi = 135°$ and $315°$

Polarization must be linear with that angle of $45°$

$\therefore AR = \infty$

2-39. $I_g = \dfrac{2}{(50+1+73)+j(25+42.5)} = \dfrac{2}{124+j67.5}$

$= (12.442 - j\,6.7724)\times 10^{-3} = 14.166\times 10^{-3}\angle -28.56^\circ$




(a). $P_s = \frac{1}{2}\,\mathrm{Re}\,(V_g \cdot I_g^*) = \mathrm{Re}\,(12.442 + j\,6.7724)\times 10^{-3} = 12.442\times 10^{-3}\ \mathrm{W}$

(b). $P_r = \frac{1}{2}|I_g|^2 R_L = 7.325\times 10^{-3}\ \mathrm{W}$

(c). $P_L = \frac{1}{2}|I_g|^2 R_L = 0.1003\times 10^{-3}\ \mathrm{W}$

The remaining supplied power is dissipated as heat in the internal resistor of the generator, or

$P_g = \frac{1}{2}|I_g|^2 R_g = 5.0169\times 10^{-3}\ \mathrm{W}$

Thus $P_r + P_L + P_g = (7.325 + 0.1003 + 5.0169)\times 10^{-3} = 12.4422\times 10^{-3} = P_s$

2-40. The impedance transfer equation of

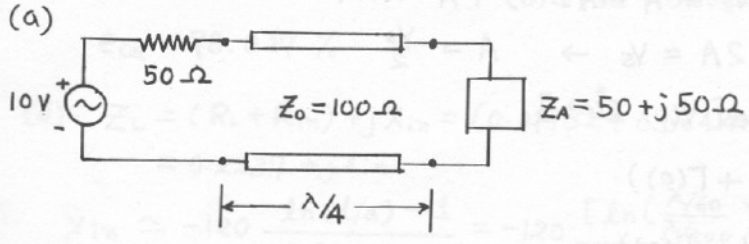
$$Z_{in} = Z_c\left[\frac{Z_L + j Z_c \tan(kl)}{Z_c + j Z_L \tan(kl)}\right]$$

reduces for $l = \lambda/2$ to $Z_{in} = Z_L$

Therefore the equivalent load impedance at the terminals of the generator is the same as that for Problem 2.39.

Thus the supplied, radiated, and dissipated powers are the same as those of Problem 2.39.

2-41. (a)




$$Z_{in} = \frac{(100)^2}{50+j50} = \frac{10\,000}{5000}(50 - j50) = 100 - j100\ \Omega$$

(Continued)

2-41. (cont'd) $I_g = \frac{10}{150-j100} = \frac{10}{180.3\angle -33.7^\circ} = 0.05546\angle 33.7^\circ$ A

(b) $P_s = \frac{1}{2} Re\{V_g I_g^*\} = \frac{1}{2} \times 10 \times 0.05546 \times \cos(33.7^\circ)$

$= 0.231$ W

(c) $P_A = \frac{1}{2} |I_g|^2 Re\{Z_{in}\} = \frac{1}{2} \times (0.05546)^2 \times 100 = 0.1538$ W

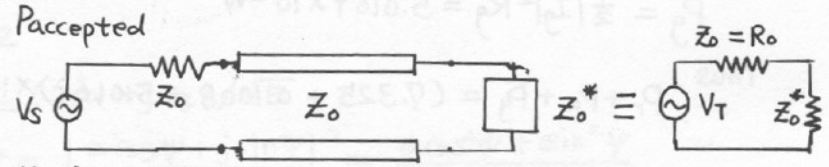
$P_{rad} = e_{cd} P_A = 0.96 \times 0.1538 = 0.148$ W

2-42. $Gain = \frac{P_{rad}}{P_{accepted}}$ Directivity

$Realized\ Gain = \frac{P_{rad}}{P_{available}}$ Directivity

$\frac{Gain}{Realized\ Gain} = \frac{P_{available}}{P_{accepted}}$

$P_{available}$




$P_{available} = \frac{1}{2} \left(\frac{V_S}{\sqrt{2}}\right)^2 \frac{1}{Z_0} = \frac{V_S^2}{4Z_0}$

$V(x) = A\,(e^{-jkx} + \Gamma(0)\, e^{jkx})$

$I(x) = \frac{A}{Z_0}(e^{-jkx} - \Gamma(0) e^{jkx})$

$V(0) = A\,(1 + \Gamma(0))$

$I(0) = \frac{A}{Z_0}(1 - \Gamma(0))$

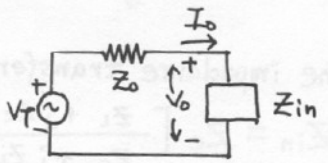



Fig 1.

From Fig 1; $-V_S + I(0) Z_0 + V(0) = 0$

$-V_S + \frac{A}{Z_0}(1 - \Gamma(0)) Z_0 + A\,(1 + \Gamma(0)) = 0$

$-V_S + A - A\Gamma(0) + A + A\Gamma(0) = 0$

$2A = V_S \rightarrow \quad A = \frac{V_S}{2}$

$P_{accepted} = Re[\, V(0)\, I^*(0)\,]$

$V(0) = \frac{V_S}{2}(1 + \Gamma(0))$

$I(0) = \frac{V_S}{2Z_0}(1 - \Gamma(0))$

$\Gamma(0) = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$

(Continued)

2-42. (Cont'd) $\Rightarrow V(0) = \frac{V_s}{2}\left(1 + \frac{Z_{in} - Z_o}{Z_{in} + Z_o}\right)$

$$= \frac{V_s}{2}\left(1 + \frac{R_{in} + jX_{in} - Z_o}{R_{in} + jX_{in} + Z_o}\right)$$

$$= \frac{V_s}{2}\left(\frac{R_{in} + jX_{in} + Z_o + R_{in} + jX_{in} - Z_o}{R_{in} + jX_{in} + Z_o}\right)$$

$$V(0) = \frac{V_s(R_{in} + jX_{in})}{R_{in} + jX_{in} + Z_o}$$

$$I(0) = \frac{V_s}{2Z_o}\left(1 - \frac{Z_{in} - Z_o}{Z_{in} + Z_o}\right) = \frac{V_s}{2Z_o}\left(\frac{Z_{in} + Z_o - Z_{in} + Z_o}{Z_{in} + Z_o}\right)$$

$$I(0) = \frac{V_s}{Z_{in} + Z_o} = \frac{V_s}{R_{in} + jX_{in} + Z_o}$$

$$\mathrm{Re}\left[V(0)\,I^*(0)\right] = \mathrm{Re}\left[\frac{V_s R_{in} + jV_s X_{in}}{R_{in} + Z_o + jX_{in}} \times \frac{V_s}{R_{in} + Z_o - jX_{in}}\right]$$

$$P_{accepted} = \mathrm{Re}\left(\frac{V_s^2(R_{in} + jX_{in})}{(R_{in} + Z_o)^2 + X_{in}^2}\right) = \frac{V_s^2 R_{in}}{(R_{in} + Z_o)^2 + X_{in}^2}$$

$$\frac{\text{Gain}}{\text{Realized Gain}} = \frac{\frac{V_s^2}{4Z_o}}{\frac{V_s^2 R_{in}}{(R_{in} + Z_o)^2 + X_{in}^2}} = \frac{(R_{in} + Z_o)^2 + X_{in}^2}{4Z_o R_{in}}$$

2-43. (a) $R_L = R_{hf}$ (2-90b) $= \frac{\ell}{C}\sqrt{\frac{\omega\mu_o}{2\sigma}}$

$$= \frac{\lambda/60}{2\pi(\lambda/200)} \cdot \sqrt{\frac{2\pi \times 10^9 (4\pi \times 10^{-7})}{2(5.7 \times 10^7)}} = 0.4415 \times 10^{-2} = 0.004415 \text{ (ohms)}$$

(b) $R_r$ (4-19) $= 80\pi^2\left(\frac{\ell}{\lambda}\right)^2 = 80\pi^2\left(\frac{1}{60}\right)^2 = 0.21932$

$\Rightarrow R_{in} = R_r = 0.21932$ ohms (because of assumed constant current)

(c) $e_{cd}$ (2-90) $= \frac{R_r}{R_L + R_r} = \frac{0.21932}{0.21932 + 0.004415} = 0.98027$

$e_{cd} = 98.027\ \%$

(d) $Z_L = (R_L + R_{in}) + jX_{in} = (0.21932 + 0.004415) + jX_{in}$

$= 0.2237 + jX_{in}$

$$X_{in} \simeq -120\,\frac{\ln(\ell/a) - 1}{\tan(k\ell)} = -120\,\frac{\left[\ln\left(\frac{\lambda/60}{\lambda/200}\right) - 1\right]}{\tan\left(\frac{2\pi}{\lambda}\,\frac{\lambda}{60}\right)}$$

$$= -120 \cdot \left[\frac{1.2040 - 1}{0.1051}\right] = -232.8806$$

(Continued)

2-43. (Cont'd)

$$|\Gamma| = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(0.2237 - j232.8806) - 50}{(0.2237 - j232.8806) + 50} = 0.9996$$

$$VSWR = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.9996}{1 - 0.9996} = 5,071.88 \simeq 5,072 \simeq \infty$$

2-44. Radiation Efficiency of a dipole

$$I_z(z) = I_0 \cos\left[\frac{\pi}{\ell} z'\right], \quad -\ell/2 \le z' \le \ell/2$$

$$H_\phi(r=a)\Big|_{\text{at the surface}} = \frac{I_0}{2\pi a} \cos\left[\frac{\pi}{\ell} z\right]$$



$ds = a\,d\phi\,dz \Rightarrow$ differential patch of area.

$dW \Rightarrow$ power loss into this patch.

$$dW = \frac{1}{2}|H_\phi|^2 R_s\, a\, d\phi\, dz$$

(time avs) ($\Leftarrow R_s$ = Skin Resistance)

$$dW = \left(\frac{I_0}{2\pi a}\right)^2 \frac{R_s}{2} \cos^2\left[\frac{\pi}{\ell} z\right] a\, d\phi\, dz$$

$$W\ (\text{total loss}) = \int_{-\ell/2}^{\ell/2} \int_{\phi=0}^{2\pi} \frac{I_0^2 R_s}{8\pi^2 a^2} \cos^2\left[\frac{\pi}{\ell} z\right] a\, d\phi\, dz$$

$$W = \frac{I_0^2}{8\pi^2 a^2} \cdot 2\pi a \cdot R_s \int_{-\ell/2}^{\ell/2} \cos^2\left[\frac{\pi}{\ell} z\right] dz = \frac{I_0^2 \ell \cdot R_s}{4\pi} \frac{1}{a} \cdot \frac{1}{2}$$

$$= \frac{1}{2} I_0^2 R_L$$

$$R_L = \frac{1}{2} \cdot \frac{\ell R_s}{2\pi a}$$

2-45.

$$E = \begin{cases} 1 & 0 < \theta \le 45^\circ \\ 0 & 45^\circ < \theta \le 90^\circ \\ \frac{1}{2} & 90^\circ < \theta \le 180^\circ \end{cases}$$

(a) $$U = \frac{r^2 E^2}{2\eta} = \frac{r^2 |E|^2}{\eta}, \quad U_{max} = \frac{r^2}{\eta} = \frac{1}{120\pi}$$

(continued)

2-45 (Cont'd)

$$P_{rad} = \frac{r^2}{\eta}\int_0^{2\pi} d\phi \left[ \int_0^{45^\circ} \sin\theta\, d\theta + \int_{90^\circ}^{180^\circ} \frac{1}{4}\sin\theta\, d\theta \right]$$

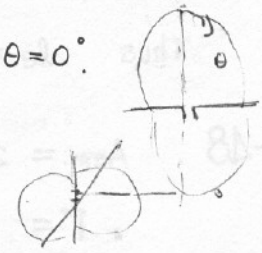
$$= \frac{r^2}{\eta}[2\pi]\left[-\cos\theta\Big|_0^{45^\circ} + \frac{1}{4}(-\cos\theta)\Big|_{90^\circ}^{180^\circ}\right]$$

$$= \frac{2r^2\pi}{\eta}\left[-\cos 45^\circ + \cos 0^\circ - \frac{1}{4}\cos 180^\circ + \frac{1}{4}\cos 90^\circ\right]$$

$$P_{rad} = 0.54289\,\frac{2\pi r^2}{\eta}$$

$$D = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi \frac{r^2}{\eta}}{0.54289\,(2\pi)r^2/\eta} = 3.684$$

(b) When the field is equal to 10 V/m, for $\theta = 0^\circ$.




$$\Rightarrow E = \begin{cases} 10 \text{ V/m} & 0 < \theta \le 45^\circ \\ 0 & 45^\circ < \theta \le 90^\circ \\ \frac{1}{2}\times 10 \text{ V/m} & 90^\circ < \theta \le 180^\circ \end{cases}$$

$$P_{rad} = \frac{r^2}{\eta}\left[\int_0^{2\pi}\left\{\int_0^{45^\circ}|E|^2\sin\theta\, d\theta + \int_{90^\circ}^{180^\circ}|E|^2\sin\theta\, d\theta\right\} d\phi\right]$$

$$P_{rad} = r^2\,(0.54289)\left(\frac{2\pi}{\eta}\right)|10|^2 = 36{,}193$$

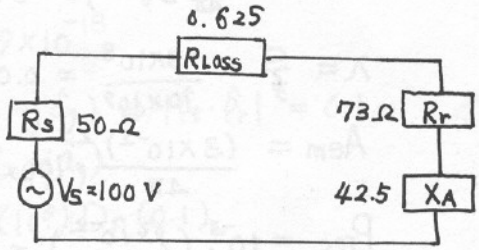
$$P_{rad} = \frac{1}{2}|I|^2 R_r = |I_{rms}|^2 R_r$$

$$\Rightarrow R_r = \frac{36{,}193}{|I_{rms}|^2} = \frac{36{,}193}{25} = 1{,}447.72$$

2-46 $f = 150\text{MHz}$, $\lambda = 2\text{ m}$

$\Rightarrow$ 1 m dipole is $\frac{\lambda}{2}$ in electrical length

$\Rightarrow R_r = 73\,\Omega$, $Z_{in} = 73 + j42.5\,\Omega$




a. $$I_{ant} = \frac{V_s}{50 + 73 + 0.625 + j42.5} = 0.765\angle -18.97^\circ \text{ A}$$

b. $$P_{dissip} = P_{Loss} = \frac{1}{2}|I_{ant}|^2 R_{Loss} = 189\text{ mW}$$

c. $$P_{rad} = \frac{1}{2}|I_{ant}|^2 R_r = 21.36\text{ W}$$

d. $$e_{cd} = \frac{R_r}{R_r + R_{Loss}} = \frac{73 \times 100}{73 + 0.625} = 99\,\%$$

2-47. Using equation (2-94), the effective aperture of an antenna can be written as

$$A_e = \frac{|V_T|^2 \cdot R_T}{2 W_i |Z_t|^2}, \quad \text{where } W_i = |E|^2/2\eta$$

Defining the effective length $l_e$ as $V_T = E \cdot l_e$ reduces $A_e$ to

$$A_e = \frac{\eta R_T l_e^2}{|Z_t|^2} \Rightarrow l_e = \sqrt{\frac{A_e |Z_t|^2}{\eta R_T}}$$

For maximum power transfer and lossless antenna ($R_L = 0$)

$$X_A = -X_T, \quad R_r = R_T \Rightarrow |Z_t| = 2R_r = 2R_T$$

Thus $$l_e = \sqrt{\frac{4A_{em} \cdot R_T^2}{\eta R_T}} = 2\sqrt{\frac{A_{em} R_T}{\eta}} = 2 \cdot \sqrt{\frac{A_{em} R_r}{\eta}}$$

2-48. $$A_{em} = 2.147 = \left(\frac{\lambda^2}{4\pi}\right) \cdot e_{cd} \cdot (1 - |\Gamma|^2) \cdot |\hat{\rho}_w \cdot \hat{\rho}_a|^2 \cdot D_o$$

$$\Gamma = \frac{75 - 50}{75 + 50} = 0.2 \; ; \quad \lambda = \frac{3 \times 10^8}{100 \times 10^6} = 3\,m$$

$$\therefore D_o = \frac{2.147}{\frac{3^2}{4\pi}[(1 - (0.2)^2]} = 3.125$$

2-49. $W_i = 10^{-3}$ W/m²

$$A_{em} = \frac{\lambda^2}{4\pi} D_o, \quad D_o = 20\,dB = 10 \log_{10} X \Rightarrow X = 100$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10 \times 10^9} = 0.03\,m = 3 \times 10^{-2}\,m$$

$$A_{em} = \frac{(3 \times 10^{-2})^2}{4\pi} \cdot 100 = \frac{9 \times 10^{-4}}{4\pi} \cdot (100) = 0.716 \times 10^{-2} = 7.16 \times 10^{-3}$$

$$P_{rec} = 10^{-3} \cdot \left(\frac{9 \times 10^{-2}}{4\pi}\right) = \frac{9 \times 10^{-5}}{4\pi} = 0.716 \times 10^{-5} = 7.16 \times 10^{-6} \text{ Watts}$$

$P_{rec} = 7.16 \times 10^{-6}$ watts.

2-50 $A_{em} = \frac{\lambda^2}{4\pi} e_t D_o = \frac{\lambda^2}{4\pi} G_o$

a. $G_o = 14.8\ dB \Rightarrow G_o\ (\text{power ratio}) = 10^{1.48} = 30.2$

$f = 8.2\ GHz \Rightarrow \lambda = 3.6585\ cm$

$A_{em} = \frac{(3.6585)^2}{4\pi}(30.2) = 32.167\ cm^2$

The physical aperture is equal to $A_p = 5.5(7.4) = 40.7\ cm^2$

b. $G_o = 16.5\ dB \Rightarrow G_o\ (\text{power ratio}) = 10^{1.65} = 44.668$

$f = 10.3\ GHz \Rightarrow \lambda = 2.912\ cm$

$A_{em} = \frac{(2.912)^2}{4\pi}(44.668) = 30.142\ cm^2$

C. $G_o = 18.0\ dB \Rightarrow G_o\ (\text{power ratio}) = 10^{1.8} = 63.096$

$f = 12.4\ GHz \Rightarrow \lambda = 2.419\ cm$

$A_{em} = \frac{(2.419)^2}{4\pi}(63.096) = 29.389\ cm^2$

2-51 $G_{ain} = 30\ dB$, $f = 2\ GHz$, $P_{rad} = 5W$

Receiving antenna VSWR = 2, efficiency = 95%

$\underline{E}_R = (2\hat{a}_x + j\hat{a}_y) F_R(\theta, \phi)$, Use Friis transmission formula (2-118)

$P_r = P_t e_{cdt} e_{cdr} (1 - |\Gamma_t|^2)(1 - |\Gamma_r|^2) \left(\frac{\lambda}{4\pi R}\right)^2 D_t(\theta_t, \phi_t) D_r(\theta_r, \phi_r) \cdot PLF$

$P_r = 10^{-14} W$, $e_{cdt} = 1$ (we assume that), $e_{cdr} = 0.95$, $1 - |\Gamma_t|^2 = 1$

Since VSWR = 2 $\Rightarrow |\Gamma_r| = \left|\frac{VSWR - 1}{VSWR + 1}\right| = \frac{2-1}{2+1} = \frac{1}{3}$, $(1 - |\Gamma_r|^2) = 8/9$

$\lambda = \frac{3 \times 10^8}{2 \times 10^9} = 0.15\ m$, $R = 4000 \times 10^3\ m$,

hence $\left(\frac{\lambda}{4\pi R}\right)^2 = \left(\frac{0.15}{4\pi \cdot 4000 \times 10^3}\right)^2 = 8.9 \times 10^{-18}$

$D_t = 30\ dB = 10^3$, PLF $\Rightarrow \begin{cases} \rho_t = \frac{1}{\sqrt{2}}(\hat{a}_x - j\hat{a}_y) \\ \rho_r = \frac{1}{\sqrt{5}}(2\hat{a}_x + j\hat{a}_y) \end{cases} \Rightarrow |\hat{\rho}_t \cdot \hat{\rho}_r|^2 = 0.1$

$\Rightarrow 10^{-14} = 5(1)(0.95)(1)\left(\frac{8}{9}\right)(8.9 \times 10^{-18})(10^3) D_r (0.1)$

$D_r = 2.661$

Hence $A_{em} = \frac{\lambda^2}{4\pi} 2.661 = 0.00476\ m^2$

2-52. $U(\theta,\phi) = \begin{cases} \cos^4(\theta), & 0^\circ \le \theta \le 90^\circ \\ 0, & 90^\circ \le \theta \le 180^\circ \end{cases} \Big\} \; 0^\circ \le \phi \le 360^\circ$

$A_{em} = \frac{\lambda^2}{4\pi} D_o$

$D_o = \frac{4\pi U_{max}}{P_{rad}}$

$P_{rad} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta \, d\theta d\phi = 2\pi \int_0^{\pi/2} \cos^4(\theta)\sin\theta d\theta = 2\pi\left[-\frac{\cos^5\theta}{5}\right]_0^{\pi/2}$

$P_{rad} = 2\pi(-0+\frac{1}{5}) = \frac{2\pi}{5}$,

$D_o = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi(1)}{2\pi/5} = 10$

$A_{em} = \frac{\lambda^2}{4\pi} D_o = \frac{\lambda^2}{4\pi}\cdot 10 = \frac{10\lambda^2}{4\pi}$, $\lambda = \frac{3\times10^8}{10^{10}} = 3\times10^{-2} = 0.03\,m$

$A_{em} = \frac{10(0.03)^2}{4\pi} = \frac{10\cdot(3\times10^{-2})^2}{4\pi} = \frac{10\cdot(9\times10^{-4})}{4\pi} = 7.16197\times10^{-4}$

$A_{em} = 7.16197\times10^{-4}$

2-53. 1 status mile = 1609.3 meter, 22,300 (status miles) $= 3.588739\times10^7 m$

a. $P_i = \frac{P_{rad}}{4\pi R^2} = \frac{8\times10^{-14}}{4\pi\times(3.58874)} = 4.943\times10^{-16}$ Watts/$m^2$.

b. $A_{em} = \frac{\lambda^2}{4\pi} D_o$, ($\leftarrow D_o = 60\,dB, = 10^6$) ($\leftarrow \lambda = 0.15\,m$)

$A_{em} = \frac{(0.15)^2}{4\pi}\cdot 10^6 = 1790.493\ m^2$

$P_{received} = A_{em}\cdot P_i = (1790.493)\cdot(4.943\times10^{-16})$

$= 8.85\times10^{-13}$ watts.

2-54 $A_{em} = 0.7162\ m^2$

$A_{em} = \left(\frac{\lambda}{4\pi}\right)^2 e_{cd}(1-|\Gamma|^2)\,|\hat{\rho}_w\cdot\hat{\rho}_a|\cdot D_o$ (polarization factor $= 1$)

$D_o = \frac{A_{em}}{\left(\frac{\lambda}{4\pi}\right)^2(1-|\Gamma|^2)}$, $\Gamma = \frac{75-50}{75+50} = 0.2$, $\lambda = \frac{3\times10^8}{100\times10^6} = 3m$

$D_o = \frac{0.7162}{\frac{3^2}{4\pi}(1-|0.2|^2)}$

$D_o = 1.0417$

2-55. $P_r = W_i A_{em} = W_i e_{cd}(1-|\Gamma|^2)\left(\frac{\lambda^2}{4\pi}\right) D_o |\hat{\rho}_w \cdot \hat{\rho}_a|^2$

$W_i = 5\ W/m^2$, $e_{cd} = 1$ (lossless), $\Gamma = \frac{Z_{in}-Z_o}{Z_{in}+Z_o} = \frac{73-50}{73+50} = 0.187$

$\lambda = \frac{3\times10^8}{10\times10^6} = 30\ m$, $D_o = 2.156\,dB = 1.643$, $PLF = 1$

$P_r = (5)(1)(1-(0.187)^2)\left(\frac{30^2}{4\pi}\right)(1.643)(1) = 567.78$ Watts $= P_r$

$P_r = 567.78$ Watts.

2-56. $\frac{P_r}{P_t} = \left(\frac{\lambda}{4\pi R}\right)^2 G_{or} G_{ot}$, $G_{or} = G_{ot} = 16.3 \Rightarrow G_o$ (power ratio) $= 42.66$

$f = 10\ GHz \Rightarrow \lambda = 0.03$ meters.

$P_t = 200$ m watts $= 0.2$ Watts

a. $R = 5\ m$ : $P_r = \left[\frac{0.03}{4\pi(5)}\right]^2 (42.66)^2 (0.2) = 82.9\ \mu$ Watts

b. $R = 50\ m$ : $P_r = 0.829\ \mu$ watts

c. $R = 500\ m$ : $P_r = 8.29$ n watts

The VSWR was not needed because the gain was given.

257. $\frac{P_r}{P_t} = |\hat{\rho}_t \cdot \hat{\rho}_r|^2 \left(\frac{\lambda}{4\pi R}\right)^2 G_{ot}\ G_{or}$

$G_{ot} = 20\,dB \Rightarrow G_{ot}$ (power ratio) $= 10^2 = 100$

$G_{or} = 15\,dB \Rightarrow G_{or}$ (power ratio) $= 10^{1.5} = 31.623$

$f = 1\ GHz \Rightarrow \lambda = 0.3$ meters

$R = 1\times10^3$ meters

a. For $|\hat{\rho}_t \cdot \hat{\rho}_r|^2 = 1$

$P_r = \left(\frac{0.3}{4\pi\times10^3}\right)^2 (100)(31.623)(150\times10^{-3}) = 270.344\ \mu$ Watts

b. When transmitting antennas is circularly polarized and receiving antenna is linearly polarized, the PLF is equal to

$|\hat{\rho}_t \cdot \hat{\rho}_r|^2 = \left|\left(\frac{\hat{a}_x \pm j\hat{a}_y}{\sqrt{2}}\right)\cdot \hat{a}_x\right|^2 = \frac{1}{2}$

Thus $P_r = \frac{1}{2}(270.344\times10^{-6}) = 135.172\times10^{-6} = 135.172\ \mu$Watts

2-58. Lossless : $e_{cd} = 1$, polarization matched : $|\hat{\rho}_w \cdot \hat{\rho}_a|^2 = 1$,
line matched : $(1 - |\Gamma|^2) = 1$

$D_o = 20\,dB = 10^2 = 100 = D_{or} = D_{ot}$

$P_r = P_t \left(\frac{\lambda}{4\pi R}\right)^2 D_{ot} D_{or} = 10 \left(\frac{\lambda}{4\pi \cdot 50\lambda}\right)^2 (100)(100) = 0.253$ watts

$P_r = 0.253$ watts

2-59. Lossless : $e_{cd} = 1$, PLF = 1. line matched : $(1 - |\Gamma|^2) = 1$.

$D_o = 30\,dB = 10^3 = 1000 = D_{or} = D_{ot}$

$P_r = P_t \left(\frac{\lambda}{4\pi \cdot 100\lambda}\right)^2 \cdot (1000)^2 = 20 \cdot \left(\frac{1}{4\pi}\right)^2 \cdot 100 = 12.665$ watts

2-60 $G_{or} = 20\,dB = 100$, $G_{ot} = 25\,dB = 316.23$. $\lambda = \frac{3 \times 10^8}{3 \times 10^9} = 0.1$ m.

$P_r = P_t \cdot |\hat{\rho}_t \cdot \hat{\rho}_r|^2 \cdot \left(\frac{\lambda}{4\pi R}\right)^2 \cdot G_{or} \cdot G_{ot} = 100 \cdot (1) \cdot \left(\frac{0.1}{4\pi \times 500}\right)^2 (100)(316.23)$

$= 8 \times 10^{-4}$ watts $= P_r$

2-61. $f = 10\,GHz, \rightarrow \lambda = \frac{3 \times 10^8}{10^{10}} = 0.03$ m

$G_{ot} = G_{or} = 15\,dB = 10^{1.5} = 31.62$

$R = 10\,km = 10^4 m$

$P_r \geq 10\,nW = 10^{-8}\,W$

$|\hat{\rho}_t \cdot \hat{\rho}_r|^2 = -3\,dB = \frac{1}{2}$

Friis Transmission Equation :

$$\frac{P_r}{P_t} = G_{ot} \cdot G_{or} \cdot \left(\frac{\lambda}{4\pi R}\right)^2 \cdot |\hat{\rho}_t \cdot \hat{\rho}_r|^2$$

$$= (10^{1.5})^2 \cdot \left(\frac{0.03}{4\pi \times 10^4}\right)^2 \cdot \left(\frac{1}{2}\right) = 2.85 \times 10^{-11}$$

$$P_t = \frac{P_r}{2.85 \times 10^{-11}}$$

$P_r \geq 10^{-8}\,W \rightarrow (P_t)_{min} = 351\,W$

2-62. $\frac{P_r}{P_t} = p\, e_t e_r\, D_{ot}\, D_{or} \left(\frac{\lambda}{4\pi R}\right)^2$

$= (p)(e_{rt}\cdot e_{cdt})(e_{rr}\cdot e_{cdr})\left(\frac{\lambda}{4\pi R}\right)^2 \cdot D_{ot}\cdot D_{or}$

$\frac{P_r}{P_t} = (1)(e_{rt}\cdot(1))(e_{rr}\cdot(1))\left(\frac{\lambda}{4\pi R}\right)^2 \cdot D_{ot}\cdot D_{or}$

$\lambda = \frac{c}{f} = \frac{3\times10^8}{10^8} = 3\text{m}, \quad R = 10\times10^3 = 10^4.$

$\left(\frac{\lambda}{4\pi R}\right)^2 = \left(\frac{3}{4\pi\times10^4}\right)^2 = \left(\frac{3}{4\pi}\times10^{-4}\right)^2 = \left(0.2387\times10^{-4}\right)^2 = 5.699\times10^{-2}\times10^{-8}$

$\left(\frac{\lambda}{4\pi R}\right)^2 = 5.699\times10^{-10}$

$e_{rt} = e_{rr} = (1-|\Gamma|^2) = \left(1-\left|\frac{73.3-50}{73.3+50}\right|^2\right) = \left(1-\left|\frac{23.3}{123.3}\right|^2\right)$

$= (1-(0.18897)^2) = (1-0.0357) = 0.9643$

$e_{cdt} = e_{cdr} = 1$

$D_{ot} = D_{or} = 1.643$

$\frac{P_r}{P_t} = (0.9643)^2(1.643)^2(5.699\times10^{-10}) = (0.92987)(2.699)(5.699\times10^{-10})$

$= 2.51\,(5.699\times10^{-10}) = 14.305\times10^{-10}$

$P_t = \frac{P_r}{14.305\times10^{-10}} = 6.99\times10^{-2}\times10^{10}(1\times10^6) = 6.99\times10^2 = 699.$

$P_t = 699$ Watts

2-63. $\frac{P_r}{P_t} = \left(\frac{\lambda}{4\pi R}\right)^2 \cdot G_{ot}\cdot G_{or}, \quad \lambda = \frac{3\times10^8}{9\times10^9} = \frac{3\times10^8}{90\times10^8} = \frac{1}{30}$

$R = 10{,}000 \text{ meter} = \frac{10{,}000}{1/30}\lambda = 3\times10^5\lambda.$

$\frac{P_r}{P_t} = \left[\frac{\lambda}{4\pi(3\times10^5\lambda)}\right]^2 \cdot G_o^2 = \frac{10\times10^{-6}}{10} = 10^{-6}$

$G_o^2 = 10^{-6}(4\pi\times3\times10^5)^2$

$G_o = 10^{-3}(4\pi\times3\times10^5) = 12\pi\times10^2 = 1200\pi$

$G_o = 1200\pi = 3{,}769.91 = 10\log_{10}(3{,}769.91)\text{ dB}$

$G_o = 3{,}769.91 = 35.76\text{ dB}$

2-64. $\sigma = \pi a^2 = 25\pi\lambda^2$

$G_{ot} = G_{or} = 16.3\ dB \Rightarrow G_{ot}\ (\text{power ratio}) = 10^{1.63} = 42.66$

$f = 10\,GHz \Rightarrow \lambda = 0.03\ m$

$$\frac{P_r}{P_t} = \sigma \frac{G_{ot} \cdot G_{or}}{4\pi} \left( \frac{\lambda}{4\pi R_1 \cdot R_2} \right)^2$$

a. $R_1 = R_2 = 200\lambda = 6$ meters;

$$P_r = 25\pi\lambda^2 \cdot \frac{(42.66)^2}{4\pi} \cdot \left[ \frac{\lambda}{4\pi (200\lambda)^2} \right]^2 \cdot (0.2) = 9.00\ n\ watts$$

b. $R_1 = R_2 = 500\lambda = 15$ meters;

$P_r = 0.23\ n\ watts$

2-65. $$P_r = P_t\, \sigma \cdot \frac{G_{ot} \cdot G_{or}}{4\pi} \cdot \left[ \frac{\lambda}{4\pi R_1 \cdot R_2} \right]^2, \quad \lambda = \frac{3 \times 10^8}{5 \times 10^9} = 0.06\ m$$

$$P_r = 10^5 (3) \cdot \frac{150^2}{4\pi} \cdot \left[ \frac{0.06}{4\pi (10^6)} \right]^2$$

$$P_r = 1.22 \times 10^{-8}\ \text{Watts}$$

2-66. $$\frac{P_r}{P_t} = \sigma \frac{G_{or} \cdot G_{ot}}{4\pi} \cdot \left[ \frac{\lambda}{4\pi R_1 R_2} \right]^2 \Rightarrow \sigma = \frac{P_r \cdot 4\pi}{P_t\, G_{or} \cdot G_{ot}} \left[ \frac{4\pi R_1 \cdot R_2}{\lambda} \right]^2$$

$$\lambda = \frac{3 \times 10^8}{3 \times 10^8} = 1\ m$$

$$\therefore \sigma = \frac{0.1425 \times 10^{-3} (4\pi)}{1000\ (75)(75)} \left[ \frac{4\pi (500)(500)}{1} \right]^2 = 3142\ m^2 = \sigma$$

2-67.

$$\sigma = \frac{P_r \cdot 4\pi}{P_t \cdot G_{or} \cdot G_{ot}} \left[ \frac{4\pi R_1 R_2}{\lambda} \right]^2$$

$$\lambda = \frac{3 \times 10^8}{1 \times 10^8} = 3\ m$$

$$\sigma = \frac{0.01 \cdot (4\pi)}{1000\ (75)(75)} \left[ \frac{4\pi (700)(700)}{3} \right]^2 = 94,114.5\ m^2$$

$$\sigma = 94,114.5\ m^2$$

2-68. $\frac{P_r}{P_t} = |\hat{\rho}_t \cdot \hat{\rho}_r|^2 \left(\frac{\lambda}{4\pi R}\right)^2 G_{ot} G_{or}$

$G_{ot} = 20\,dB \rightarrow G_{ot} = 10^2 = 100$

$G_{or} = 15dB \rightarrow G_{or} = 10^{1.5} = 31.623$

$f = 1\,GHz \rightarrow \lambda = 0.3$ meters

$R = 10^3$ meters

a) For PLF = 1.

$P_r = \left(\frac{0.3}{4\pi \times 10^3}\right)^2 (100)(31.623)(150) = 270.344\,\mu$ Watts

b) $PLF = \left|\left(\frac{\hat{a}_x \pm j\hat{a}_y}{\sqrt{2}}\right) \cdot \hat{a}_x\right|^2 = \frac{1}{2} \rightarrow P_r = \frac{270.344}{2}\,\mu$ Watts $= 135.172\,\mu$Watts

↑ or any other linearly polarized unit vector

2-69. $\sigma = 0.85\lambda^2$

$\frac{P_r}{P_t} = \sigma \cdot \frac{G_{ot} \cdot G_{or}}{4\pi} \left(\frac{\lambda}{4\pi R_1 \cdot R_2}\right)^2 |\hat{\rho}_w \cdot \hat{\rho}_r|^2$

$\sigma = 0.85\lambda^2$, $G_{ot} = G_{or} = 15dB \Rightarrow G_{ot} = G_{or} = 31.6228$ (dimensionless)

$R_1 = R_2 = 100$ meter $\Rightarrow R_1 = R_2 = 1{,}000\lambda$

$f = 3\,GHz \Rightarrow \lambda = \frac{3\times10^8}{3\times10^9} = 0.1$ meters

$|\hat{\rho}_w \cdot \hat{\rho}_r|^2 = 1\,dB \Rightarrow |\hat{\rho}_w \cdot \hat{\rho}_r|^2 = 0.7943$

$\frac{P_r}{P_t} = 0.85\lambda^2 \cdot \frac{(31.6228)^2}{4\pi} \cdot \left(\frac{\lambda}{4\pi \times 10^6 \lambda^2}\right)^2 \cdot (0.7943)$

$= \frac{0.85\,(31.6228)^2(0.7943)}{(4\pi)^3\,(10^{12})} = 0.3402 \times 10^{-12}$

$P_r = 0.3402 \times 10^{-12}(10^2) = 0.3402 \times 10^{-10} = 34.02 \times 10^{-12}$ watts

$P_r = 34.02$ pWatts

2-70. $T_a = T_A e^{-2\alpha l} + T_o(1 - e^{-2\alpha l})$

$T_A = 5\,^\circ K$

$T_o = 72^\circ F = \frac{5}{9}(72-32) + 273 = 295.2\,^\circ K$

$-4\,dB = 20 \log_{10} e^{-\alpha} = -\alpha(20)\log_{10} e = -\alpha(20)(0.434)$

$\alpha = \frac{4}{8.68} = 0.460$ Nepers/100ft $= 0.0046$ N/ft.

a. $l = 2$ feet;

$T_a = 5e^{-2(0.0046)2} + 295.2[1 - e^{-2(0.0046)2}] = 4.91 + 5.38 = 10.29\,^\circ K$

b. $l = 100$ feet,

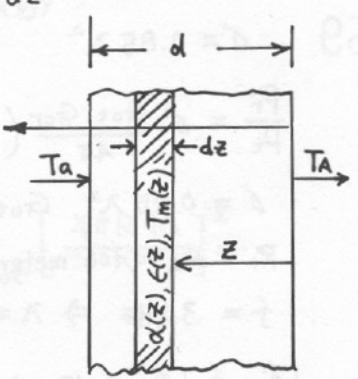
$T_a = 5e^{-2(0.0046)100} + 295.2[1 - e^{-2(0.0046)100}] = 179.72\,^\circ K$

2-71 $T_a = T_A e^{-\int_0^d 2\alpha(z)dz} + \int_0^d \epsilon(z)\, T_m(z)\, e^{-\int_z^d 2\alpha(z')dz'} dz$

If $\alpha(z) = \alpha_o = $ Constant

$T_a = T_A e^{-2\alpha_o d} + \int_0^d \epsilon(z)\, T_m(z)\, e^{-2\alpha_o(d-z)} dz$

$T_a = T_A e^{-2\alpha_o d} + e^{-2\alpha_o d} \int_0^d \epsilon(z) T_m(z) e^{+2\alpha_o z} dz$




If $T_m(z) = T_o = $ Constant and $\epsilon(z) = \epsilon_o = $ Constant

$T_a = T_A e^{-2\alpha_o d} + \epsilon_o T_o e^{-2\alpha_o d} \int_0^d e^{2\alpha_o z} dz$

$T_a = T_A e^{-2\alpha_o d} + \frac{\epsilon_o}{2\alpha_o} T_o e^{-2\alpha_o d}(e^{2\alpha_o d} - 1)$

For $\epsilon_o = 2\alpha_o$

$T_a = T_A e^{-2\alpha_o d} + T_o e^{-2\alpha_o d}(e^{2\alpha_o d} - 1)$

$= T_A e^{-2\alpha_o d} + T_o(1 - e^{-2\alpha_o d})$

3-1. If $\underline{H}_e = j\omega\epsilon \nabla \times \underline{\Pi}_e$ (1)

Maxwell's curl equation $\nabla \times \underline{E}_e = -j\omega\mu \underline{H}_e$ can be written as

$$\nabla \times \underline{E}_e = -j\omega\mu \underline{H}_e = -j\omega\mu(j\omega\epsilon \nabla \times \underline{\Pi}_e) = \omega^2\mu\epsilon \nabla \times \underline{\Pi}_e$$

or $\nabla \times (\underline{E}_e - \omega^2\mu\epsilon \underline{\Pi}_e) = \nabla \times (\underline{E}_e - k^2 \underline{\Pi}_e) = 0$ where $k^2 = \omega^2\mu\epsilon$

Letting $\underline{E}_e - k^2 \underline{\Pi}_e = -\nabla \phi_e \Rightarrow \underline{E}_e = -\nabla \phi_e + k^2 \underline{\Pi}_e$ (2)

Taking the curl of (1) and using the vector identity of Equation (3-8) leads to

$$\nabla \times \underline{H}_e = j\omega\epsilon \nabla \times \nabla \times \underline{\Pi}_e = j\omega\epsilon [\nabla(\nabla \cdot \underline{\Pi}_e) - \nabla^2 \underline{\Pi}_e] \quad (3)$$

Using Maxwell's equation

$$\nabla \times \underline{H}_e = \underline{J} + j\omega\epsilon \underline{E}_e$$

reduces (3) to

$$\underline{J} + j\omega\epsilon \underline{E}_e = j\omega\epsilon [\nabla(\nabla \cdot \underline{\Pi}_e) - \nabla^2 \underline{\Pi}_e] \quad (4)$$

Substituting (2) into (4) reduces to

$$\nabla^2 \underline{\Pi}_e + k^2 \underline{\Pi}_e = j\frac{\underline{J}}{\omega\epsilon} + [\nabla(\nabla \cdot \underline{\Pi}_e) + \nabla \phi_e] \quad (5)$$

Letting $\phi_e = -\nabla \cdot \underline{\Pi}_e$ simplifies (5) to

$$\nabla^2 \underline{\Pi}_e + k^2 \underline{\Pi}_e = j\frac{\underline{J}}{\omega\epsilon} \quad (6)$$

and (2) to

$$\underline{E}_e = \nabla(\nabla \cdot \underline{\Pi}_e) + k^2 \underline{\Pi}_e \quad (7)$$

Comparing (6) with (3-14) leads to the relation

$$\underline{\Pi}_e = -j\frac{1}{\omega\mu\epsilon} \underline{A} \quad (8)$$

3-2. If $\underline{E}_m = -j\omega\mu \nabla \times \underline{\Pi}_m$ (1)

Maxwell's curl equation $\nabla \times \underline{H}_m = j\omega\epsilon \underline{E}_m$ can be written as

$$\nabla \times \underline{H}_m = j\omega\epsilon(-j\omega\mu \nabla \times \underline{\Pi}_m) = \omega^2\mu\epsilon \nabla \times \underline{\Pi}_m$$

or $\nabla \times (\underline{H}_m - \omega^2\mu\epsilon \underline{\Pi}_m) = \nabla \times (\underline{H}_m - k^2 \underline{\Pi}_m) = 0$ where $k^2 = \omega^2\mu\epsilon$

Letting $\underline{H}_m - k^2 \underline{\Pi}_m = -\nabla \phi_m \Rightarrow \underline{H}_m = -\nabla \phi_m + k^2 \underline{\Pi}_m$ (2)

(Continued)

3-2 (Cont'd)

Taking the curl of (1) and using the vector identity of Equation (3-8) leads to

$$\nabla \times \underline{E}_m = -j\omega\mu \nabla \times \nabla \times \underline{\Pi}_m = -j\omega\mu [\nabla(\nabla \cdot \underline{\Pi}_m) - \nabla^2 \underline{\Pi}_m] \quad (3)$$

Using Maxwell's equation

$\nabla \times \underline{E}_m = -\underline{M} - j\omega\mu \underline{H}_m$ reduces (3) to

$$-M - j\omega\mu \underline{H}_m = -j\omega\mu [\nabla(\nabla \cdot \underline{\Pi}_m) - \nabla^2 \underline{\Pi}_m] \quad (4)$$

Substituting (2) into (4) reduces to

$$\nabla^2 \underline{\Pi}_m + k^2 \underline{\Pi}_m = j\frac{\underline{M}}{\omega\mu} + [\nabla(\nabla \cdot \underline{\Pi}_m) + \nabla \phi_m] \quad (5)$$

Letting $\phi_m = -\nabla \cdot \underline{\Pi}_m$ simplifies (5) to

$$\nabla^2 \underline{\Pi}_m + k^2 \underline{\Pi}_m = j\frac{\underline{M}}{\omega\mu} \quad (6)$$

and (2) to

$$\underline{H}_m = \nabla(\nabla \cdot \underline{\Pi}_m) + k^2 \underline{\Pi}_m \quad (7)$$

Comparing (6) with (3-25) leads to the relation

$$\underline{\Pi}_m = -j\frac{1}{\omega\mu\epsilon} \underline{F}$$

3-3. $$\underline{A} = \hat{a}_z A_{z1} = \hat{a}_z C_1 \frac{e^{-jkr}}{r}$$

Substituting the above into (3-34) leads to the following terms;

$$\frac{d^2 A_{z1}}{dr^2} = \frac{d}{dr}\left[\frac{d}{dr}\left(C_1 \frac{e^{-jkr}}{r}\right)\right] = C_1 \frac{d}{dr}\left[-jk\frac{e^{-jkr}}{r} - \frac{e^{-jkr}}{r^2}\right]$$

$$= C_1 \left[(-jk)^2 \frac{e^{-jkr}}{r} + jk\frac{e^{-jkr}}{r^2} - jk\frac{e^{-jkr}}{r^2} + 2\frac{e^{-jkr}}{r^3}\right]$$

$$\frac{2}{r}\frac{dA_{z1}}{dr} = \frac{2}{r} C_1 \left(-jk\frac{e^{-jkr}}{r} - \frac{e^{-jkr}}{r^2}\right)$$

$$k^2 A_{z1} = k^2 C_1 \frac{e^{-jkr}}{r}$$

The sum of the above three terms is equal to zero, and it satisfies (3-34)

The same conclusion is derived using

$$\underline{A} = \hat{a}_z A_{z2} = \hat{a}_z C_2 \frac{e^{-jkr}}{r}$$

as a solution

3-4. The solution of $\nabla^2 A_z = -\mu J_z$ can be inferred from the solution of Poisson's equation

$$\nabla^2 \phi = -\frac{\rho}{\epsilon} \tag{1}$$

for the potential $\phi$. $\rho(x';y';z')$ represents the charge density

We begin with Green's theorem

$$\int_V (\psi \nabla^2 \phi - \phi \nabla^2 \psi)\, dv' = \oint_\Sigma (\psi \nabla\phi - \phi \nabla\psi)\cdot \hat{n}\, da \tag{2}$$

where $\psi$ and $\phi$ are well behaved functions (nonsingular, continuous, and twice differentiable). For $\psi$ we select a solution of the form

$$\psi = \frac{1}{R} \tag{3}$$

where $$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \tag{3a}$$

By considering the charge at the origin of the coordinate system, it can be shown that (provided $r \neq 0$)

$$\nabla^2 \psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} = \nabla^2\left(\frac{1}{r}\right) = 0$$

Thus (2) reduces to

$$\int_V \psi \nabla^2 \phi\, dv' = -\frac{1}{\epsilon}\int_V \frac{\rho(x';y';z')}{r}\, dv' \tag{4}$$

To exclude the $r=0$ singularity of $\psi$, the observation point $x', y', z'$ is surrounded by a sphere of radius $r'$ and surface $\Sigma'$. Therefore the volume $V$ is bounded by the surfaces $\Sigma$ and $\Sigma'$, and (2) is broken into two integrals; one over $\Sigma$ and the other of $\Sigma'$. Using (4) reduces (2) to

$$-\frac{1}{\epsilon}\int_V \frac{\rho}{r}\, dv' = \oint_\Sigma (\psi \nabla\phi - \phi\nabla\psi)\cdot\hat{n}\, da + \oint_{\Sigma_0} (\psi\nabla\phi - \phi\nabla\psi)\cdot\hat{n}\, da \tag{5}$$

and

$$\oint_{\Sigma'} (\psi\nabla\phi - \phi\nabla\psi)\cdot\hat{n}\, da = \oint_{\Sigma'} \left[\frac{1}{r'}\nabla\phi - \phi(\nabla\psi)_{r=r'}\right]\cdot\hat{n}\, da$$

$$= -\frac{1}{r'}\oint_{\Sigma'} \frac{\partial\phi}{\partial r}\, da - \frac{1}{r'^2}\oint_{\Sigma'} \phi\, da \tag{5a}$$

(Continued)

3-4 (Cont'd) Since $r'$ is arbitrary, it can be chosen small enough so that $\phi$ and $\frac{\partial \phi}{\partial r}$ are essentially constant at every point on $\Sigma'$. If we make $r'$ progressively smaller, $\phi$ and its normal derivative approach their limiting values at the center (by hypothesis, both exist and are continuous functions of position). Therefore in the limit as $r' \to 0$, both can be taken outside the integral and we can write that

$$\oint_{\Sigma'} (\psi \nabla\phi - \phi \nabla\psi) \cdot \hat{n}\, da = -4\pi \phi(x,y,z) \qquad (6)$$

Since

$$\lim_{r'\to 0} \frac{1}{r'} \oint_{\Sigma'} \frac{\partial \phi}{\partial r}\, da = \lim_{r'\to 0} \frac{1}{r'} \left(\frac{\partial \phi}{\partial r}\right)_{r=r'} \oint_{\Sigma'} da = \lim_{r'\to 0} \frac{1}{r'} \left(\frac{\partial \phi}{\partial r}\right)_{r=r'} (4\pi r'^2) = 0$$

Substituting (6) into (5) reduces it to

$$\phi(x,y,z) = \frac{1}{4\pi\epsilon} \int_V \frac{\rho}{r}\, dv' + \frac{1}{4\pi} \oint_\Sigma \left[\frac{1}{r} \nabla\phi - \phi \nabla\left(\frac{1}{r}\right)\right] \cdot \hat{n}\, da \qquad (7)$$

The first term on the right side of (7) accounts for the contributions from the charges within $\Sigma$ while the second term for those outside $\Sigma$. Expansion of $\Sigma$ to include all charges makes the second term to vanish and to reduce (7) to

$$\phi(x,y,z) = \frac{1}{4\pi\epsilon} \int_V \frac{\rho(x',y',z')}{r}\, dv' \qquad (8)$$

By comparing $\nabla^2 A_z = -\mu J_z$ with (1) we can write that

$$A_z(x,y,z) = \frac{\mu}{4\pi} \int_V \frac{J_z(x',y',z')}{r}\, dv' \qquad (9)$$

For more details see D.T. Paris and F. K. Hard, Basic Electromagnetic Theory, McGraw-Hill, 1969, pp. 128-131.

For the details of the solution of (3-31) see

R.E. Collin, Field Theory of Guided Waves, McGraw-Hill, 1960, pp. 35-39. It can be shown that

$$A_z = \frac{\mu}{4\pi} \int_V J_z(x',y',z') \frac{e^{-jkr}}{r}\, dv$$

Because of the length of the derivation, it will not be repeated here.

3-5. $\underline{A} \simeq [\hat{a}_r A'_r(\theta,\phi) + \hat{a}_\theta A'_\theta(\theta,\phi) + \hat{a}_\theta A'_\phi(\theta,\phi)] \frac{e^{-jkr}}{r}$

$\underline{E} = -j\omega\underline{A} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla\cdot\underline{A})$

$\psi = \nabla\cdot\underline{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$

$\psi = \nabla\cdot\underline{A} = -jk\frac{e^{-jkr}}{r} + \frac{e^{-jkr}}{r^2}[\cdots\cdots] + \frac{e^{-jkr}}{r^3}[\cdots\cdots] + \cdots$

$\nabla(\nabla\cdot\underline{A}) = \nabla\psi = \hat{a}_r\frac{\partial\psi}{\partial r} + \hat{a}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{a}_\phi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$

$= \hat{a}_r\{\frac{1}{r}[-\omega^2\mu\epsilon e^{-jkr}A'_r(\theta,\phi)] + \frac{1}{r^2}[\cdots] + \frac{1}{r^3}[\cdots] + \cdots\}$

$+ \hat{a}_\theta\{\frac{1}{r}(0) + \frac{1}{r^2}[\cdots] + \frac{1}{r^3}[\cdots] + \cdots\}$

$+ \hat{a}_\phi\{\frac{1}{r}(0) + \frac{1}{r^2}[\cdots] + \frac{1}{r^3}[\cdots] + \cdots\}$

Therefore

$\underline{E} = -j\omega\underline{A} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla\cdot\underline{A})$

$\underline{E} \simeq -j\omega[\hat{a}_r A'_r + \hat{a}_\theta A'_\theta + \hat{a}_\phi A'_\phi]\frac{e^{-jkr}}{r}$

$-j\frac{1}{\omega\mu\epsilon}\{\hat{a}_r[\omega^2\mu\epsilon\frac{e^{-jkr}}{r} + \frac{1}{r^2}(\cdots) + \frac{1}{r^3}(\cdots) + \cdots]$

$+ \hat{a}_\theta[\frac{1}{r}(0) + \frac{1}{r^2}(\cdots) + \frac{1}{r^3}(\cdots) + \cdots]$

$+ \hat{a}_\phi[\frac{1}{r}(0) + \frac{1}{r^2}(\cdots) + \frac{1}{r^3}(\cdots) + \cdots]\}$

or $\underline{E} \simeq \frac{1}{r}\{-j\omega e^{-jkr}[\hat{a}_r(0) + \hat{a}_\theta A'_\theta + \hat{a}_\phi A'_\phi]\} + \frac{1}{r^2}[\cdots] + \frac{1}{r^3}[\cdots]\cdots\}$

In a similiar manner, it can be shown that

$\underline{H} = \frac{1}{\mu}\nabla\times\underline{A} = \frac{1}{r}\{j\frac{\omega}{\eta}e^{-jkr}[\hat{a}_r(0) + \hat{a}_\theta A'_\phi - \hat{a}_\phi A'_\theta]\} + \frac{1}{r^2}[\cdots] + \frac{1}{r^3}[\cdots] + \cdots$

3-6. Let us assume that within a linear and isotropic medium, but not necessarily homogeneous, there exist two sets of sources $\underline{J}_1, \underline{M}_1$ and $\underline{J}_2, \underline{M}_2$ which are allowed to radiate simultaneously or individually inside the same medium at the same frequency and produce $\underline{E}_1, \underline{H}_1$ and $\underline{E}_2, \underline{H}_2$, respectively. For the fields to be valid, they must satisfy Maxwell's equations

(Continued)

3-6. (Cont'd)

$$\nabla \times \underline{E}_1 = -\dot{z}\underline{H}_1 - \underline{M}_1 \quad (1)$$

$$\nabla \times \underline{H}_1 = \dot{y}\underline{E}_1 + \underline{J}_1 \quad (2)$$

$$\nabla \times \underline{E}_2 = -\dot{z}\underline{H}_2 - \underline{M}_2 \quad (3)$$

$$\nabla \times \underline{H}_2 = \dot{y}\underline{E}_2 + \underline{J}_2 \quad (4)$$

where $\dot{z} = j\omega(\mu' - j\mu'')$ (5)

$\dot{y} = \sigma + j\omega(\epsilon' - j\epsilon'')$ (6)

If we dot multiply (1) by $\underline{H}_2$ and (4) by $\underline{E}_1$, we can write

$$\underline{H}_2 \cdot \nabla \times \underline{E}_1 = -\dot{z}\underline{H}_2 \cdot \underline{H}_1 - \underline{H}_2 \cdot \underline{M}_1 \quad (7)$$

$$\underline{E}_1 \cdot \nabla \times \underline{H}_2 = \dot{y}\underline{E}_1 \cdot \underline{E}_2 + \underline{E}_1 \cdot \underline{J}_2 \quad (8)$$

Subtracting (7) from (8) reduces to

$$\underline{E}_1 \cdot \nabla \times \underline{H}_2 - \underline{H}_2 \cdot \nabla \times \underline{E}_1 = \dot{y}\underline{E}_1 \cdot \underline{E}_2 + \dot{z}\underline{H}_2 \cdot \underline{H}_1 + \underline{E}_1 \cdot \underline{J}_2 + \underline{H}_2 \cdot \underline{M}_1 \quad (9)$$

which by using the vector identity

$$\nabla \cdot (\underline{A} \times \underline{B}) = \underline{B} \cdot (\nabla \times \underline{A}) - \underline{A} \cdot (\nabla \times \underline{B}) \quad (10)$$

can be written as

$$\nabla \cdot (\underline{H}_2 \times \underline{E}_1) = -\nabla \cdot (\underline{E}_1 \times \underline{H}_2) = \dot{y}\underline{E}_1 \cdot \underline{E}_2 + \dot{z}\underline{H}_2 \cdot \underline{H}_1 + \underline{E}_1 \cdot \underline{J}_2 + \underline{H}_2 \cdot \underline{M}_1 \quad (11)$$

In a similar manner, if we dot multiply (2) by $\underline{E}_2$ and (3) by $\underline{H}_1$, we can write

$$\underline{E}_2 \cdot \nabla \times \underline{H}_1 = \dot{y}\underline{E}_2 \cdot \underline{E}_1 + \underline{E}_2 \cdot \underline{J}_1 \quad (12)$$

$$\underline{H}_1 \cdot \nabla \times \underline{E}_2 = -\dot{z}\underline{H}_1 \cdot \underline{H}_2 - \underline{M}_1 \cdot \underline{M}_2 \quad (13)$$

Substracting (13) from (12) leads to

$$\underline{E}_2 \cdot \nabla \times \underline{H}_1 - \underline{H}_1 \cdot \nabla \times \underline{E}_2 = \dot{y}\underline{E}_2 \cdot \underline{E}_1 + \dot{z}\underline{H}_1 \cdot \underline{H}_2 + \underline{E}_2 \cdot \underline{J}_1 + \underline{H}_1 \cdot \underline{M}_2 \quad (14)$$

which by using (10) can be written as

$$\nabla \cdot (\underline{H}_1 \times \underline{E}_2) = -\nabla \cdot (\underline{E}_2 \times \underline{H}_1) = \dot{y}\underline{E}_2 \cdot \underline{E}_1 + \dot{z}\underline{H}_1 \cdot \underline{H}_2 + \underline{E}_2 \cdot \underline{J}_1 + \underline{H}_1 \cdot \underline{M}_2 \quad (15)$$

Substracting (15) from (11) leads to

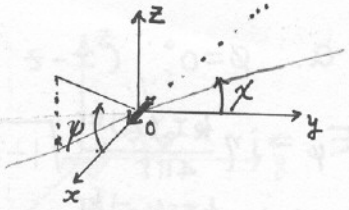
$$-\nabla \cdot (\underline{E}_1 \times \underline{H}_2 - \underline{E}_2 \times \underline{H}_1) = \underline{E}_1 \cdot \underline{J}_2 + \underline{H}_2 \cdot \underline{M}_1 - \underline{E}_2 \cdot \underline{J}_1 - \underline{H}_1 \cdot \underline{M}_2 \quad (16)$$

Which is known as the Lorentz Reciprocity Theorem in differential form. Taking the volume integral of both sides of (16) and using the divergence theorem on the left side, we can write (16) as

$$-\oint\!\!\oint (\underline{E}_1 \times \underline{H}_2 - \underline{E}_2 \times \underline{H}_1) \cdot d\underline{s} = \iiint_v (\underline{E}_1 \cdot \underline{J}_2 + \underline{H}_2 \cdot \underline{M}_1 - \underline{E}_2 \cdot \underline{J}_1 - \underline{H}_1 \cdot \underline{M}_2)\, dv \quad (17)$$

which is known as the Lorentz Reciprocity Theorem in integral form.

4-1. a.




$$\sin\psi = \sqrt{1-\cos^2\psi} = \sqrt{1-|\hat{a}_z\cdot\hat{a}_r|^2}$$

$$= \sqrt{1-(\sin\theta\cdot\cos\phi)^2}$$

In far-zone fields

$$E_\psi \simeq j\eta\frac{kI_0\cdot l e^{-jkr}}{4\pi r}\cdot\sin\psi = j\eta\frac{kI_0 l e^{-jkr}}{4\pi r}\cdot\sqrt{1-(\sin\theta\cdot\cos\phi)^2}$$

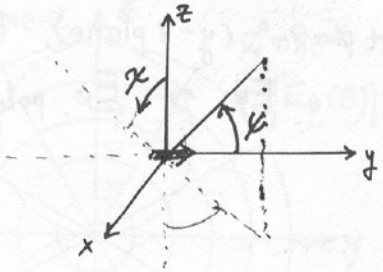
$$H_\chi \simeq j\frac{kI_0 l e^{-jkr}}{4\pi r}\cdot\sin\psi = \frac{E_\psi}{\eta}$$

b. $U = U_0(1-\sin^2\theta\cos^2\phi)$

$$\therefore P_{rad} = U_0\int_0^{2\pi}\int_0^{\pi}(1-\sin^2\theta\cdot\cos^2\phi)\cdot\sin\theta\,d\theta\,d\phi = U_0\cdot\frac{8\pi}{3}$$

$$D_0 = \frac{4\pi\cdot U_0}{U_0\cdot\frac{8\pi}{3}} = \frac{3}{2} = 1.5$$

4-2.




a. $\sin\psi = \sqrt{1-\cos^2\psi} = \sqrt{1-|\hat{a}_y\cdot\hat{a}_r|^2}$

$$= \sqrt{1-\sin^2\theta\cdot\sin^2\phi}$$

In far-zone fields

$$E_\psi \simeq j\eta\frac{kI_0 l e^{-jkr}}{4\pi r}\cdot\sin\psi = j\eta\frac{kI_0 l e^{-jkr}}{4\pi r}\cdot\sqrt{1-\sin^2\theta\cdot\sin^2\phi}$$

$$H_\chi \simeq \frac{E_\psi}{\eta} \simeq j\frac{kI_0 e^{-jkr}\cdot l}{4\pi r}\sqrt{1-\sin^2\theta\sin^2\phi}$$

b. $U = U_0(1-\sin^2\theta\sin^2\phi)$

$$P_{rad} = U_0\int_0^{2\pi}\int_0^{\pi}(1-\sin^2\theta\cdot\sin^2\phi)\sin\theta\,d\theta\,d\phi = U_0\int_0^{2\pi}\left[\int_0^{\pi}\sin\theta - \sin^3\theta\cdot\sin^2\phi\,d\theta\right]d\phi$$

$$= U_0\left[\int_0^{2\pi}2\,d\phi - \frac{4}{3}\int_0^{2\pi}\sin^2\phi\,d\phi\right] = U_0\left[4\pi - \frac{4}{3}\pi\right] = \frac{8}{3}\pi\cdot U_0$$

$$D_0 = \frac{4\pi\cdot U_0}{U_0\cdot\frac{8\pi}{3}} = \frac{3}{2} = 1.5$$

4-3.

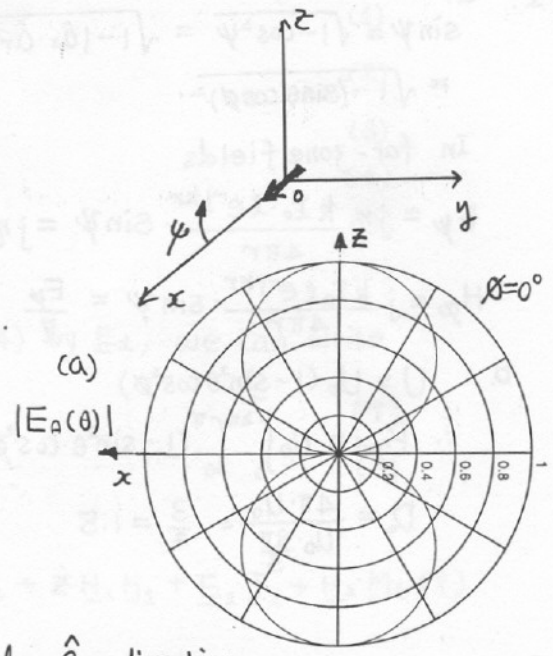
a. $\phi = 0°$ ($x$-$z$ plane)

$$E_\psi = j\eta \frac{kI_0 l e^{-jkr}}{4\pi r} \sqrt{1-\sin^2\theta}$$

$$\simeq j\eta \frac{kI_0 l e^{-jkr}}{4\pi r} \cdot \cos\theta$$

At $\phi = 0°$, $E_\psi$ has only $\hat{a}_\theta$ direction.

$E_\psi \sim E_\theta$ polarization



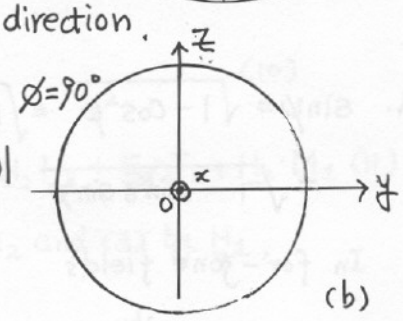

(a) $|E_\theta(\theta)|$

b. $\phi = 90°$ ($y$-$z$ plane)

$$E_\psi \simeq j\eta \frac{kI_0 l e^{-jkr}}{4\pi r} \cdot 1$$

At $\phi = 90°$, ($y$-$z$ plane), $E_\psi$ has only $\hat{a}_\phi$ direction.

$E_\psi \sim E_\phi$ polarization




(b) $\phi = 90°$ $|E_\phi(\theta)|$

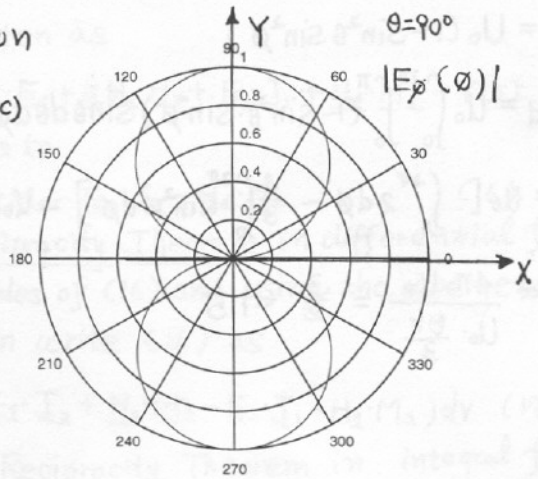
(b)

c. $\theta = 90°$ ($x$-$y$ plane)

$$E_\psi = j\eta \frac{kI_0 l e^{-jkr}}{4\pi r} \sqrt{1-\cos^2\phi} = j\eta \frac{kI_0 l e^{-jkr}}{4\pi r} \cdot \sin\phi$$

At $\theta = 90°$. ($x$, $y$), $E_\psi$ has only $\hat{a}_\phi$ direction.

$E_\psi \sim E_\phi$ polarization



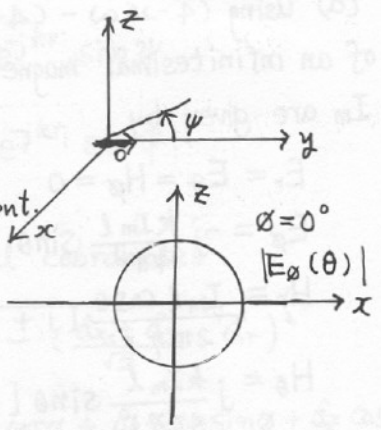

(c) $\theta = 90°$ $|E_\phi(\phi)|$

4-4. a. $\phi = 0°$ ( $x-z$ plane)

$$E_\psi = j\eta \frac{kI_0 e^{-jkr}}{4\pi r} \cdot 1$$

At $\phi = 0°$, $E_\psi$ direction has only $\hat{a}_\phi$ component.

$E_\psi \rightsquigarrow E_\phi$ polarization



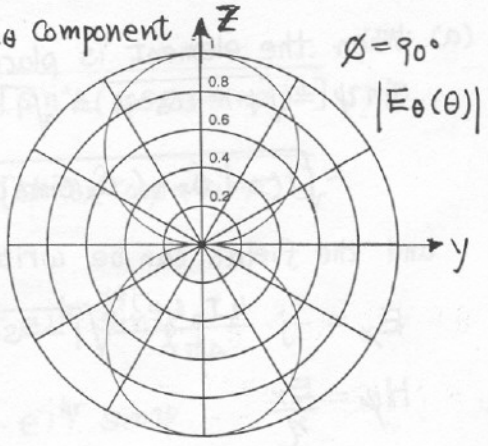

b. $\phi = 90°$ ($y-z$ plane)

$$E_\psi = j\eta \frac{k_0 I_0 l e^{-jkr}}{4\pi r} \cdot \sqrt{1-\sin^2\theta}$$

$$= j\eta \frac{k_0 I_0 l e^{-jkr}}{4\pi r} \cdot \cos\theta .$$

At $\phi = 90°$, $E_\psi$ direction has only $\hat{a}_\theta$ component

$E_\psi \rightsquigarrow E_\theta$ polarization

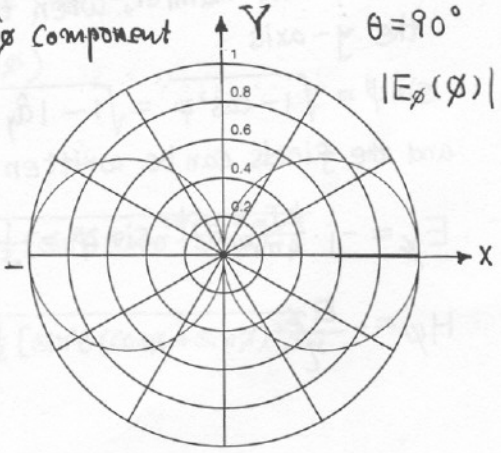



c. $\theta = 90°$ ( $x$, $y$) plane.

$$E_\psi = j\eta \frac{k_0 I_0 l e^{-jkr}}{4\pi r} \cdot \cos\phi$$

At $\theta = 90°$, $E_\psi$ direction has only $\hat{a}_\phi$ component

$E_\psi \rightsquigarrow E_\phi$ polarization



4-5. (a) Using (4-26a) - (4-26c) and the duality of Table 3.2, the fields of an infinitesimal magnetic dipole of length $\ell$ and magnetic current $I_m$ are given by

$$E_r = E_\theta = H_\phi = 0$$

$$E_\phi = -j\frac{kI_m\ell}{4\pi r}\sin\theta\left[1+\frac{1}{jkr}\right]e^{-jkr}$$

$$H_r = \frac{I_m\ell\cos\theta}{2\pi\eta r^2}\left[1+\frac{1}{jkr}\right]e^{-jkr}$$

$$H_\theta = j\frac{kI_m\ell}{4\pi\eta r}\sin\theta\left[1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right]e^{-jkr}$$

(b) Since the pattern of the magnetic dipole is the same as that of the electric, the directivities are also identical and equal to

$$D_o = \frac{3}{2}\ (\text{dimensionless}) = 1.761\ \text{dB}$$

4-6. (a) When the element is placed along the x-axis

$$\sin\psi = \sqrt{1-\cos^2\psi} = \sqrt{1-|\hat{a}_x\cdot\hat{a}_r|^2}$$

$$= \sqrt{1-|\hat{a}_x\cdot(\hat{a}_x\sin\theta\cos\phi+\hat{a}_y\sin\theta\sin\phi+\hat{a}_z\cos\theta)|^2}$$

and the fields can be written as

$$E_x = -j\cdot\frac{kI_m\ell e^{-jkr}}{4\pi r}\sqrt{1-\sin^2\theta\cos^2\phi} = -j\cdot\frac{kI_m\ell e^{-jkr}}{4\pi r}\sin\psi$$

$$H_\psi = -\frac{E_x}{\eta}$$

(b) In a similar manner, when the element is placed along the y-axis

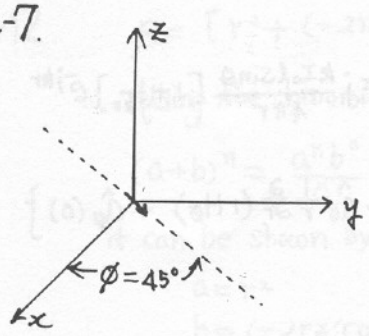
$$\sin\psi = \sqrt{1-\cos^2\psi} = \sqrt{1-|\hat{a}_y\cdot\hat{a}_r|^2} = \sqrt{1-\sin^2\theta\sin^2\phi}$$

and the fields can be written as

$$E_\chi = -j\cdot\frac{kI_m\ell}{4\pi r}e^{-jkr}\sin\psi = -j\cdot\frac{kI_m\ell}{4\pi r}e^{-jkr}\sqrt{1-\sin^2\theta\sin^2\phi}$$

$$H_\psi = -\frac{E_\chi}{\eta}$$

4-7.




$$E_\psi = j\eta \frac{kI_0 l}{4\pi r} e^{-jkr} \sin\psi$$

$$H_\chi = j \frac{kI_0 l}{4\pi r} e^{-jkr} \sin\psi$$

Convert $\psi$ to spherical coordinates

$$\sin\psi = \sqrt{1-\cos^2\psi} = \sqrt{1-\left(\frac{\hat{a}_x+\hat{a}_y}{\sqrt{2}}\cdot\hat{a}_r\right)^2}$$

$$\leftarrow \frac{\hat{a}_x+\hat{a}_y}{\sqrt{2}}\cdot\hat{a}_r = \left(\frac{\hat{a}_x}{\sqrt{2}}+\frac{\hat{a}_y}{\sqrt{2}}\right)\cdot\left(\hat{a}_x\sin\theta\cos\phi + \hat{a}_y\sin\theta\sin\phi + \hat{a}_z\cos\theta\right)$$

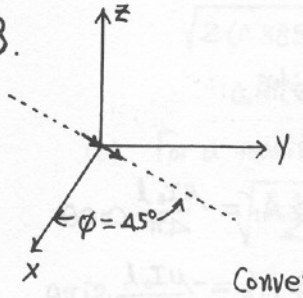
$$= \frac{\sin\theta\cos\phi}{\sqrt{2}} + \frac{\sin\theta\sin\phi}{\sqrt{2}} = \frac{1}{\sqrt{2}}\sin\theta(\cos\phi+\sin\phi)$$

Thus

$$E_\psi = j\eta \frac{kI_0 l}{4\pi r} e^{-jkr} \sqrt{1-\tfrac{1}{2}[\sin^2\theta(\cos\phi+\sin\phi)^2]}$$

$$H_\chi = j \frac{kI_0 l}{4\pi r} e^{-jkr} \sqrt{1-\tfrac{1}{2}[\sin^2\theta(\cos\phi+\sin\phi)^2]}$$

4-8.




$$H_\psi = j \frac{kI_m l}{4\pi\eta r} e^{-jkr} \sin\psi$$

$$E_\chi = -j \frac{kI_m l}{4\pi r} e^{-jkr} \sin\psi$$

Convert $\psi$ to spherical coordinates

$$\sin\psi = \frac{1}{\sqrt{2}}\sin\theta(\sin\phi+\cos\phi)$$

Thus

$$H_\psi = j \frac{kI_m l}{4\pi\eta r} e^{jkr} \sqrt{1-\tfrac{1}{2}[\sin^2\theta(\cos\phi+\sin\phi)^2]}$$

$$E_\chi = -j \frac{kI_m l}{4\pi r} e^{jkr} \sqrt{1-\tfrac{1}{2}[\sin^2\theta(\cos\phi+\sin\phi)^2]}$$

4-9.

$$\underline{E} = \frac{1}{j\omega\epsilon}\nabla\times\underline{H} \qquad \text{where } H_r = H_\theta = 0,\ H_\phi = j\frac{kI_0 l\sin\theta}{4\pi r}\left[1+\frac{1}{jkr}\right]e^{-jkr}$$

Since $\underline{H}$ is not a function of $\phi$

$$\underline{E} = \frac{1}{j\omega\epsilon}\nabla\times\underline{H} = \frac{1}{j\omega\epsilon}\left\{\hat{a}_r\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(H_\phi\sin\theta) - \hat{a}_\theta\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) + \hat{a}_\phi(0)\right\}$$

which reduces using the $H_\phi$ from above to

$$E_r = \eta\frac{I_0 l\cos\theta}{2\pi r^2}\left[1+\frac{1}{jkr}\right]e^{-jkr}$$

$$E_\theta = j\eta\frac{kI_0 l\sin\theta}{4\pi r}\left[1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right]e^{-jkr}$$

$$E_\phi = 0$$

4-10.

$$\underline{W}_{ave} = \frac{1}{2}Re\left[\underline{E}\times\underline{H}^*\right] = \frac{1}{2}Re\left[\hat{a}_\theta E_\theta\times\hat{a}_\phi H_\phi^*\right]$$

$$\underline{W}_{ave} = \hat{a}_r W_r = \frac{1}{2}Re\left[\hat{a}_\theta E_\theta\times\hat{a}_\phi\frac{E_\theta^*}{\eta}\right] = \hat{a}_r\frac{1}{2\eta}Re\left(|E_\theta|^2\right) = \hat{a}_r\frac{|E_\theta|^2}{2\eta}$$

$$W_r = \left[\frac{\eta}{2}\left|\frac{kI_0 l}{4\pi}\right|^2\right]\frac{\sin^2\theta}{r^2} = W_0\frac{\sin^2\theta}{r^2}, \quad \text{where } W_0 = \frac{\eta}{2}\left|\frac{kI_0 l}{4\pi}\right|^2$$

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi}\underline{W}_{ave}\cdot\hat{a}_r r^2\sin\theta\,d\theta\,d\phi = 2\pi W_0\int_0^{\pi}\sin^3\theta\,d\theta = 2\pi W_0\left(\frac{4}{3}\right)$$

$$P_{rad} = \frac{8\pi}{3}W_0 = \eta\left(\frac{\pi}{3}\right)\left|\frac{I_0 l}{\lambda}\right|^2$$

4-11.

$$\underline{A} = \hat{a}_z A_z = \hat{a}_z\frac{\mu I_0 l}{4\pi r}e^{-jkr} \Rightarrow A_z = \frac{\mu_0 I_0 l}{4\pi r}e^{-jkr}$$

Using (4-6a) - (4-6c)

$$A_r = A_z\cos\theta = \frac{\mu I_0 l e^{-jkr}}{4\pi r}\cos\theta = A_r'(\theta)\frac{e^{-jkr}}{r} \Rightarrow A_r' = \frac{\mu I_0 l}{4\pi}\cos\theta$$

$$A_\theta = -A_z\sin\theta = -\frac{\mu I_0 l e^{-jkr}}{4\pi r}\sin\theta = A_\theta'(\theta)\frac{e^{-jkr}}{r} \Rightarrow A_\theta' = -\frac{\mu I_0 l}{4\pi}\sin\theta$$

$$A_\phi = 0 \qquad \Rightarrow A_\phi' = 0$$

Substituting these into (3-57) and (3-57a) reduces to

$$E_r = 0$$

$$E_\theta = -j\omega\frac{e^{-jkr}}{r}A_\theta' = j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\sin\theta = j\eta\frac{kI_0 l e^{-jkr}}{4\pi r}\sin\theta$$

$$E_\phi = -j\omega\frac{e^{-jkr}}{r}A_\phi' = 0$$

$$H_r = 0$$

$$H_\theta = -j\frac{\omega}{\eta}\frac{e^{-jkr}}{r}A_\phi' = 0$$

$$H_\phi = -j\frac{\omega}{\eta}\frac{e^{-jkr}}{r}A_\theta' = j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi\eta r}\cdot\sin\theta = j\frac{kI_0 l e^{-jkr}}{4\pi r}\sin\theta$$

which are identical to (4-26a) - (4-26c)

4-12. $R = [r^2 + (-2rz'\cos\theta + z'^2)]^{1/2} = r\left[1 + \left(\frac{-2rz'\cos\theta + z'^2}{r^2}\right)\right]^{1/2}$

Using the binomial Expansion of

$$(a+b)^n = \frac{a^n b^0}{0!} + \frac{n a^{n-1} b^1}{1!} + (n)(n-1)\frac{a^{n-2} b^2}{2!} + (n)(n-1)(n-2)\frac{a^{n-3} b^3}{3!} + \cdots$$

it can be shown by letting

$a = r^2$

$b = (-2rz'\cos\theta + z'^2)$

$n = \frac{1}{2}$

that $R = r - z'\cos\theta + \frac{1}{r}\left(\frac{z'^2}{2}\sin^2\theta\right) + \frac{1}{r^2}\left(\frac{z'^3}{2}\cos\theta\cdot\sin^2\theta\right)$

$$+ \frac{1}{r^3}\left[\frac{z'^4}{8}(-1 + 6\cos^2\theta - 5\cos^4\theta)\right] + \cdots$$

Therefore the fifth term of (4-41) is

$$\frac{1}{r^3}\left[\frac{z'^4}{8}(-1 + 6\cos^2\theta - 5\cos^4\theta)\right]$$

4-13. For maximum phase error of $\pi/8$ radians

$$0.62\sqrt{D^3/\lambda} \leq r \leq 2D^2/\lambda$$

(a) For a maximum phase error of $\pi/16$ radians

$$\sqrt{2(0.385)}\sqrt{D^3/\lambda} \leq r \leq 4D^2/\lambda$$

$$0.8775\sqrt{D^3/\lambda} \leq r \leq 4D^2/\lambda$$

(b) For a maximum phase error of $\pi/4$ radians

$$\sqrt{\frac{0.385}{2}}\sqrt{D^3/\lambda} \leq r \leq D^2/\lambda$$

$$0.43875\sqrt{D^3/\lambda} \leq r \leq D^2/\lambda$$

(c) For a maximum phase error of 18° radians

$$18^\circ \rightarrow \frac{\pi}{10} \text{ radians}$$

$$\sqrt{1.25(0.385)}\sqrt{D^3/\lambda} \leq r \leq (1.25)\cdot 2\, D^2/\lambda$$

$$0.6937\sqrt{D^3/\lambda} \leq r \leq 2.5\, D^2/\lambda$$

(d) For a maximum phase error of 15° radians

$$15^\circ \rightarrow \frac{\pi}{12} \text{ radians}.$$

$$\sqrt{1.5(0.385)}\sqrt{D^3/\lambda} \leq r \leq (1.5)\cdot 2\cdot D^2/\lambda$$

$$0.7599\sqrt{D^3/\lambda} \leq r \leq 3\cdot D^2/\lambda$$

4-14. $\ell = 5\lambda_o \Rightarrow z'|_{max} = 2.5\lambda$

a. Far-Field (Fraunhofer)

$$r = \frac{2\ell^2}{\lambda} = \frac{2(5\lambda)^2}{\lambda} = \frac{2(25\lambda^2)}{\lambda} = 50\lambda$$

$$\Delta\phi_e = \frac{k}{r}\left(\frac{z'^2}{2}\sin^2\theta\right)\Big|_{\substack{\theta=30^\circ,\ z'=2.5\lambda,\\ r=50\lambda}} \simeq \frac{\frac{2\pi}{\lambda}}{50\lambda}\left[\frac{(2.5)^2\lambda^2}{2}\,\frac{1}{4}\right] = 0.0982\ \text{rads} = 5.625^\circ$$

b. Fresnel

$$r = 0.62\sqrt{\ell^3/\lambda} = 0.62\sqrt{(5\lambda)^3/\lambda} = 0.62\lambda\cdot\sqrt{125} = 6.9318\lambda$$

$$\Delta\phi_e = \frac{k}{r^2}\left(\frac{z'^3}{2}\cos\theta\sin^2\theta\right)\Big|_{\substack{\theta=30^\circ\\ z'=2.5\lambda\\ r=6.9318\lambda}} = \frac{2\pi}{\lambda}\frac{(2.5\lambda)^3}{(6.9318\lambda)^2\cdot 2}\left(\cos 30^\circ\right)\left(\sin 30^\circ\right)^2$$

$$\Delta\phi_e = \frac{\pi(2.5)^3}{(6.9318)^2}\,(0.866)(0.5)^2 = 0.2212\ \text{rads} = 12.6724^\circ$$

4-15. $$\underline{A} = \hat{a}_z\frac{\mu I_o}{4\pi}\int_0^\ell e^{-jkz'}\frac{e^{-jkR}}{R}dz' \cong \hat{a}_z\frac{\mu I_o}{4\pi r}e^{-jkr}\int_0^\ell e^{-jk(1-\cos\theta)z'}dz'$$

$$A_z \simeq \frac{\mu I_o e^{-jkr}}{4\pi r}\int_0^\ell \frac{e^{-jk(1-\cos\theta)z'}\,d[-jk(1-\cos\theta)z']}{-jk(1-\cos\theta)}$$

$$A_z \simeq \frac{\mu I_o e^{-jkr}}{4\pi r}\left[\frac{e^{-jk(1-\cos\theta)z'}}{-jk(1-\cos\theta)}\right]_0^\ell = \frac{\mu I_o \ell e^{-jkr}}{4\pi r}e^{-jz}\frac{\sin(z)}{z}$$

where $z = \frac{k\ell}{2}(1-\cos\theta)$

(a) $\left.\begin{aligned} A_r &= A_z\cos\theta\\ A_\theta &= -A_z\sin\theta\\ A_\phi &= 0\end{aligned}\right\} \Rightarrow$ For far-field $\Rightarrow \left\{\begin{aligned} E_\theta &\simeq -j\omega A_\theta\\ E_\phi &\simeq -j\omega A_\phi\\ E_r &\simeq 0\end{aligned}\right.$

Therefore $E_r \simeq 0 \simeq H_r$, $E_\theta \simeq j\frac{\omega\mu I_o \ell e^{-jkr}}{4\pi r}e^{-jz}\frac{\sin(z)}{z}\sin\theta$

$E_\phi = 0 = H_\theta$, $H_\phi \simeq \frac{E_\theta}{\eta}$

(b) $$\underline{W}_{ave} = \underline{W}_{rad} = \frac{1}{2}\mathrm{Re}[\underline{E}\times\underline{H}^*] = \frac{1}{2\eta}|E_\theta|^2$$

$$= \frac{1}{2\eta}\left|\frac{\omega\mu I_o \ell}{4\pi r}\cdot\frac{\sin(z)}{z}\cdot\sin\theta\right|^2$$

4-16. (a) $\underline{A} = \frac{\mu}{4\pi}\int_{-\infty}^{\infty} \underline{I}(z')\frac{e^{-jkR}}{R}dz' = \hat{a}_z \frac{\mu I_0}{4\pi}\int_{-\infty}^{\infty}\frac{e^{-jkR}}{R}dz'$

where $R = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\Big|_{x'=y'=0} = \sqrt{x^2+y^2+(z-z')^2}$

Making a change of variable of the form,

$z - z' = -\rho$ , $dz' = d\rho$

reduces the potential to

$$A_z = \frac{\mu I_0}{4\pi}\int_{-\infty}^{\infty}\frac{e^{-jk\sqrt{\rho^2+\rho^2}}}{\sqrt{\rho^2+\rho^2}}d\rho \qquad \text{where } \rho^2 = x^2+y^2$$

Using $\int_{-\infty}^{\infty}\frac{e^{-j\beta\sqrt{b^2+t^2}}}{\sqrt{b^2+t^2}}dt = -j\pi H_0^{(2)}(b\beta)$

We can write the potential as

$$A_z = -j\frac{\mu I_0}{4}H_0^{(2)}(k\rho) = -j\frac{\mu I_0}{4}H_0^{(2)}(k\sqrt{x^2+y^2})$$

(b) $\underline{H} = \frac{1}{\mu}\nabla\times\underline{A}$ and $\underline{E} = \frac{1}{j\omega\epsilon}\nabla\times\underline{H}$

Since $A_\rho = A_\phi = 0$, in cylindrical coordinates

$$\underline{H} = \frac{1}{\mu}\nabla\times\underline{A} = \frac{1}{\mu}\left(-\hat{a}_\phi\frac{\partial A_z}{\partial\rho}\right) = \hat{a}_\phi\, j\frac{I_0}{4}\frac{\partial}{\partial\rho}H_0^{(2)}(k\rho)$$

Using Equation (V-19), we can write the $\underline{H}$-field as

$$\underline{H} = \hat{a}_\phi H_\phi = -\hat{a}_\phi\, j\frac{kI_0}{4}H_1^{(2)}(k\rho)$$

where $H_1^{(2)}(k\rho)$ is the Hankel function of the second kind of order one and argument $k\rho$.

The electric field can be obtained using

$$\underline{E} = \frac{1}{j\omega\epsilon}\nabla\times\underline{H} = \hat{a}_z\frac{1}{j\omega\epsilon}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\phi)\right] = \hat{a}_z\frac{1}{j\omega\epsilon}\left(\frac{\partial H_\phi}{\partial\rho}+\frac{H_\phi}{\rho}\right)$$

$$= \hat{a}_z\frac{1}{j\omega\epsilon}\left[-j\frac{kI_0}{4}\frac{\partial}{\partial\rho}H_1^{(2)}(k\rho) - j\frac{kI_0}{4\rho}H_1^{(2)}(k\rho)\right]$$

Since $\frac{\partial}{\partial\rho}H_1^{(2)}(k\rho) = k\,H_0^{(2)}(k\rho) - \frac{1}{\rho}H_1^{(2)}(k\rho)$ by using V-18

then

$$\underline{E} = \hat{a}_z\left[-\frac{kI_0}{4\omega\epsilon}k\,H_0^{(2)}(k\rho)\right] = -\hat{a}_z\,\eta\,\frac{I_0 k}{4}H_0^{(2)}(k\rho)$$

4-17. $P_{rad} = \eta \frac{|I_0|^2}{4\pi} I_{int}$

where $I_{int} = \int_0^{\pi} \frac{[\cos(\frac{kl}{2}\cos\theta) - \cos(\frac{kl}{2})]^2}{\sin\theta} d\theta$

which can also be written as

$$I_{int} = 2\int_0^{\pi/2} \frac{[\cos^2(\frac{kl}{2}\cos\theta) + \cos^2(\frac{kl}{2}) - 2\cos(\frac{kl}{2}\cos\theta)\cos(\frac{kl}{2})]}{\sin\theta} d\theta$$

Letting $\begin{cases} \cos\theta = u \\ -\sin\theta d\theta = du \Rightarrow d\theta = -\frac{du}{\sin\theta} \end{cases}$

reduces $I_{int}$ to

$$I_{int} = 2\int_0^1 \frac{[\cos^2(\frac{kl}{2}u) + \cos^2(\frac{kl}{2}) - 2\cos(\frac{kl}{2}u)\cos(\frac{kl}{2})]}{1-u^2} du$$

$$= \int_{-1}^{1} \frac{[\cos^2(\frac{kl}{2}u) + \cos^2(\frac{kl}{2}) - 2\cos(\frac{kl}{2}u)\cos(\frac{kl}{2})]}{(1+u)} du$$

$$= \frac{1}{2}\int_{-1}^{1} \frac{[1+\cos(klu) + 1 + \cos(kl)}{(1+u)} du - \int_{-1}^{1} \frac{\cos[\frac{kl}{2}(1+u)] + \cos[\frac{kl}{2}(1-u)]}{1+u} du$$

Making another change of variable of the form

$$(1+u)kl = v \Rightarrow du = \frac{dv}{kl}$$

we can write that

$$I_{int} = \frac{1}{2}\int_0^{2kl} \frac{2 + \cos(kl) + \cos(klv)}{v} dv - \int_0^{kl} \frac{\cos v}{v} dv - \int_{-1}^{1} \frac{\cos[\frac{kl}{2}(1-u)]}{1+u} du$$

$$= \int_0^{kl} \frac{1+\cos(kl) - \cos(v)}{v} dv + \frac{1}{2}\int_0^{2kl} \frac{-\cos(kl) + \cos(v-kl)}{v} dv$$

$$- \int_0^1 \frac{\cos[kl(1-v)]}{v} dv$$

provided $v = \frac{1+u}{2}$

If $z = klv$,

$$I_{int} = \int_0^{kl} \frac{1+\cos(kl) - \cos(v)}{v} dv$$

$$+ \frac{1}{2}\int_0^{2kl} \frac{-\cos(kl) + \cos(v)\cos(kl) + \sin(v)\sin(kl)}{v} dv - \int_0^{kl} \frac{\cos(kl)\cos(z) + \sin(kl)\sin(z)}{z} dz$$

(Continued)

4-17 (Cont'd)

$$I_{int} = [1+\cos(kl)]\int_0^{kl}\frac{1-\cos v}{v}\,dv - 2\int_0^{kl}\frac{\sin(v)\sin(kl)}{v}\,dv$$
$$+ \sin(kl)\int_0^{2kl}\frac{\sin v}{v}\,dv - \cos(kl)\int_0^{2kl}\frac{1-\cos v}{v}\,dv$$

which reduces to

$$I_{int} = \{ C + \ln(kl) - C_i(kl) + \tfrac{1}{2}\sin(kl)[S_i(2kl) - 2S_i(kl)]$$
$$+ \tfrac{1}{2}\cos(kl)[C + \ln(\tfrac{kl}{2}) + C_i(2kl) - 2C_i(kl)]\}$$

where $C = 0.5772$

and $P_{rad} = \eta\frac{|I_0|^2}{4\pi} I_{int}$ is identical to (4-68)

From (4-88) $P_{rad} = \eta\frac{|I_0|^2}{4\pi}\int_0^{\pi}\frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin\theta}\,d\theta$

Letting $\left.\begin{array}{l} u = \cos\theta \\ du = -\sin\theta\,d\theta \end{array}\right\} \Rightarrow \sin^2\theta = 1-\cos^2\theta = 1-u^2.$

We can write

$$P_{rad} = -\eta\frac{|I_0|^2}{2\pi}\int_1^0\frac{\cos^2(\frac{\pi}{2}u)}{1-u^2}\,du = \eta\frac{|I_0|^2}{2\pi}\int_0^1\frac{\cos^2(\frac{\pi}{2}u)}{1-u^2}\,du$$

which can also be written as

$$P_{rad} = \eta\frac{|I_0|^2}{4\pi}\left[\int_0^1\frac{\cos^2(\frac{\pi}{2}u)}{1-u}\,du + \int_0^1\frac{\cos^2(\frac{\pi}{2}u)}{1+u}\,du\right]$$

Making another change of variable of the form

$\left.\begin{array}{l} v = 1-u \\ dv = -du \end{array}\right\}$ for the first integral, $\left.\begin{array}{l} v = 1+u \\ dv = du \end{array}\right\}$ for the second integral

we can write $P_{rad}$ as

$$P_{rad} = \eta\frac{|I_0|^2}{4\pi}\left\{\int_0^1\frac{\sin^2(\frac{\pi}{2}v)}{v}dv + \int_1^2\frac{\sin^2(\frac{\pi}{2}v)}{v}dv\right\} = \eta\frac{|I_0|^2}{4\pi}\int_0^2\frac{\sin^2(\frac{\pi}{2}v)}{v}dv$$

Using the half-angle identity $\left\langle \sin^2(\frac{\pi}{2}v) = \frac{1-\cos(\pi v)}{2}\right\rangle$

reduces $P_{rad}$ to $P_{rad} = \eta\frac{|I_0|^2}{8\pi}\int_0^2\frac{[1-\cos(\pi v)]}{v}dv$

By letting $y = \pi v$, $dy = \pi\,dv$

we can write $P_{rad}$ as

$$P_{rad} = \eta\frac{|I_0|^2}{8\pi}\int_0^{2\pi}\left[\frac{1-\cos(y)}{y}\right]dy = \eta\frac{|I_0|^2}{8\pi}C_{in}(2\pi)$$

4-18. (a) $\underline{I}_z(z') = \begin{cases} I_0\left(1+\frac{2}{\ell}z'\right), & -\frac{\ell}{2} < z' < 0 \\ I_0\left(1-\frac{2}{\ell}z'\right), & 0 < z' < \ell/2 \end{cases}$

$$\underline{A}(\underline{r}) \cong \hat{a}_z \frac{\mu}{4\pi}\frac{e^{jkr}}{r}\int_{-\ell/2}^{\ell/2} I_z(z')\, e^{jk\hat{a}_r\cdot\bar{r}'}\, dz'$$

$$= \hat{a}_z \frac{\mu}{4\pi}\frac{e^{jkr}}{r}\int_{-\ell/2}^{\ell/2}\left(1-2\frac{|z'|}{\ell}\right)e^{jkz'\cos\theta}\, dz'$$

$$= \hat{a}_z \frac{\mu}{4\pi}\frac{e^{jkr}}{r}\left\{\ell\,\frac{\sin\left(\frac{k\ell}{2}\cos\theta\right)}{\left(\frac{k\ell}{2}\cos\theta\right)} - 2\int_{-\ell/2}^{\ell/2}\frac{|z'|}{\ell}e^{jkz'\cos\theta}\, dz'\right\}$$

$$\int_{-\ell/2}^{\ell/2}\frac{|z'|}{\ell}e^{jkz'\cos\theta}dz' = \int_0^{\ell/2}\frac{z'}{\ell}e^{jkz'\cos\theta}dz' - \int_{-\ell/2}^{0}\frac{z'}{\ell}e^{jkz'\cos\theta}\, dz'\Big\}$$

$$= \int_0^{\ell/2}\frac{z'}{\ell}e^{jkz'\cos\theta}dz' + \int_0^{\ell/2}\frac{z'}{\ell}e^{-jkz'\cos\theta}dz'$$

$$= 2\int_0^{\ell/2}\frac{z'}{\ell}\cos[kz'\cos\theta]\,dz' = \frac{\ell}{2}\int_0^1 \xi\cos\left[\frac{k\ell}{2}\xi\cos\theta\right]d\xi$$

$$= \frac{\ell}{2}\left\{\frac{\sin\left(\frac{k\ell}{2}\cos\theta\right)}{\frac{k\ell}{2}\cos\theta} + \frac{\cos\left(\frac{k\ell}{2}\cos\theta\right)-1}{\left(\frac{k\ell}{2}\cos\theta\right)^2}\right\}$$

$$\therefore \underline{A}(\vec{r}) = \hat{a}_z\frac{\mu\ell}{4\pi}\frac{e^{jkr}}{r}\left\{\frac{1-\cos\left(\frac{k\ell}{2}\cos\theta\right)}{\left(\frac{k\ell}{2}\cos\theta\right)^2}\right\}$$

$$A_\theta = \hat{a}_\theta\cdot\underline{A} = -\frac{\mu\ell}{4\pi}\frac{e^{jkr}}{r}\sin\theta\left\{\frac{1-\cos\left(\frac{k\ell}{2}\cos\theta\right)}{\left(\frac{k\ell}{2}\cos\theta\right)^2}\right\}$$

$$A_\phi = \hat{a}_\phi\cdot\underline{A} = 0$$

In the far-zone $E_r \simeq 0$

$$E_\theta \simeq j\omega\mu\frac{\ell}{4\pi}\frac{e^{jkr}}{r}\sin\theta\left\{\frac{1-\cos\left(\frac{k\ell}{2}\cos\theta\right)}{\left(\frac{k\ell}{2}\cos\theta\right)^2}\right\}$$

$E_\phi \simeq 0$

$H_r \simeq 0$

$H_\theta \simeq 0$

$H_\phi \simeq E_\theta/\eta$

(continued)

4-18. (Cont'd)

(b) From (4-58 a).

$$E_\theta = j\eta \frac{ke^{-jkr}}{4\pi r} \sin\theta \left[ \int_{-l/2}^{l/2} I(z') e^{jkz'\cos\theta} dz' \right]$$

$$E_\theta = j\eta \frac{ke^{-jkr}}{4\pi r} \sin\theta \cdot I_0 \int_{-l/2}^{l/2} \cos\left(\frac{\pi z'}{l}\right) e^{jkz'\cos\theta} dz'$$

let $a = jk\cos\theta$ and $b = \frac{\pi}{l}$, use following integral formula

$$\int \cos bz\, e^{az} dz = \frac{e^{az}(a\cos bz + b\sin bz)}{a^2 + b^2}$$

$$E_\theta = j\eta \frac{ke^{jkr}}{4\pi r} \sin\theta\ I_0 \left\{ \frac{e^{jkz'\cdot\cos\theta}}{\left(\frac{\pi}{l}\right)^2 - k^2\cos^2\theta} \left[ jk\cos\theta \cos\frac{\pi z'}{l} + \frac{\pi}{l}\sin\frac{\pi z'}{l} \right] \right\}_{-l/2}^{l/2}$$

$$= j\eta \frac{ke^{-jkr}}{4\pi r} \sin\theta \cdot I_0 \left[ \frac{e^{jk\,l/2\cos\theta}}{\left(\frac{\pi}{l}\right)^2 - k^2\cos^2\theta} \frac{\pi}{l} + \frac{e^{-jk\,l/2\cos\theta}}{\left(\frac{\pi}{l}\right)^2 - k^2\cos^2\theta} \frac{\pi}{l} \right]$$

$$E_\theta = j\eta \frac{I_0 k e^{-jkr}}{4\pi r} \sin\theta \cdot \frac{\pi}{l} \cdot \frac{2\cos\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{\pi}{l}\right)^2 - k^2\cos^2\theta} = j\eta \frac{I_0 e^{-jkr}}{2\pi r} \cdot \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

$$H_\phi = j \frac{I_0 k e^{jkr}}{4\pi r} \sin\theta \cdot \frac{\pi}{l} \cdot \frac{2\cos\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{\pi}{l}\right)^2 - k^2\cos^2\theta} = j \frac{I_0 e^{-jkr}}{2\pi r} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

(c) $$E_\theta = j\eta \frac{ke^{jkr}}{4\pi r} \sin\theta \left[ \int_{-l/2}^{l/2} I_0 \cos^2\left(\frac{\pi}{l} z'\right) e^{jkz'\cos\theta} dz' \right.$$

let $a = jk\cos\theta$ and $b = \frac{\pi}{l}$, use the following integral formula

$$\int \cos^2 bz \cdot e^{az} dz = \frac{e^{az}}{2a} + \frac{e^{az}}{a^2 + 4b^2} \left( \frac{a}{2}\cos 2bz + b\sin 2bz \right)$$

$$E_\theta = j\eta \frac{ke^{-jkr}}{4\pi r} \sin\theta \cdot I_0 \left\{ \frac{e^{jkz'\cos\theta}}{2jk\cos\theta} + \frac{e^{jkz'\cos\theta}}{\left(\frac{2\pi}{l}\right)^2 - k^2\cos^2\theta} \left( \frac{jk\cos\theta}{2} \cdot \cos\frac{2\pi}{l} z' + \frac{\pi}{l}\sin\frac{2\pi}{l} z' \right) \right\}_{-l/2}^{l/2}$$

$$= j\eta \frac{ke^{-jkr}}{4\pi r} \sin\theta \cdot I_0 \left\{ \frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{k\cos\theta} + k\cos\theta \frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{2\pi}{l}\right)^2 - k^2\cos^2\theta} \right\}$$

$$H_\phi = j \frac{ke^{-jkr}}{4\pi r} \sin\theta\ I_0 \left\{ \frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{k\cos\theta} + k\cos\theta \frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{2\pi}{l}\right)^2 - k^2\cos^2\theta} \right\}$$

419. $VSWR = \frac{1+|\Gamma|}{1-|\Gamma|}$, $\Gamma = \frac{R_{in}-z_o}{R_{in}+z_o}$, $R_{in} = \frac{R_r}{\sin^2(\frac{kl}{2})}$, $z_o = 50$

(a) $l = \lambda/4$, $kl/2 = \pi/4$, $kl = \pi/2$, $2kl = \pi$

$$R_r = 60\left\{C + \ln(\pi/2) - C_i(\pi/2) + \tfrac{1}{2}\sin(\tfrac{\pi}{2})[S_i(\pi) - 2S_i(\tfrac{\pi}{2})]\right\}$$

$$R_r = 60\left\{0.5772 + 0.45158 - 0.470 + \tfrac{1}{2}[1.85 - 2(1.3698)]\right\} = 6.8388$$

$$R_{in} = \frac{R_r}{\sin^2(\frac{kl}{2})} = \frac{6.8388}{\sin^2(\frac{\pi}{4})} = 13.6776$$

$$\Gamma = \frac{13.6776-50}{13.6776+50} = 0.5704 \Rightarrow VSWR = \frac{1+0.5704}{1-0.5704} = 3.6555$$

(b) $l = \lambda/2$ : $kl/2 = \pi/2$, $kl = \pi$, $2kl = 2\pi$

$$R_r = 60\left\{C + \ln(\pi) - C_i(\pi) + \tfrac{1}{2}\cos(\pi)[C + \ln(\tfrac{\pi}{2}) + C_i(2\pi) - 2C_i(\pi)]\right\}$$

$$= 60\left\{0.5772 + 1.14473 - 0.059 - \tfrac{1}{2}[0.5772 + 0.45158 - 0.0227 - 2(0.059)]\right\}$$

$$R_r = 73.13 \Rightarrow R_{in} = \frac{R_r}{\sin^2(\frac{kl}{2})} = \frac{73.13}{\sin^2(\frac{\pi}{2})} = 73.13$$

$$\Gamma = \frac{73.13-50}{73.13+50} = 0.18785 \Rightarrow VSWR = \frac{1+0.18785}{1-0.18785} = 1.4626$$

(c) $l = 3\lambda/4$ ; $kl/2 = 3\pi/4$, $kl = 3\pi/2$, $2kl = 3\pi$

$$R_r = 60\left\{0.5772 + \ln(\tfrac{3\pi}{2}) - C_i(\tfrac{3\pi}{2}) + \tfrac{1}{2}\sin(\tfrac{3\pi}{2})[S_i(3\pi) - 2S_i(\tfrac{3\pi}{2})]\right\}$$

$$= 60\left\{0.5772 + 1.5502 - (-0.19839) - \tfrac{1}{2}[1.67473 - 2(1.611)]\right\}$$

$$R_r = 185.965 \Rightarrow R_{in} = \frac{185.965}{\sin^2(3\pi/4)} = 371.93$$

$$\Gamma = \frac{371.93-50}{371.93+50} = 0.7630 \Rightarrow VSWR = \frac{1+0.7630}{1-0.7630} = 7.4386$$

(Continued)

4-19. (Cont'd)

(d) $\ell = \lambda$ ; $k\ell/2 = \pi$ , $k\ell = 2\pi$; $2k\ell = 4\pi$

$$R_r = 60\{0.5772 + \ln(2\pi) - C_i(2\pi) + \tfrac{1}{2}\cos(2\pi)[0.5772 + \ln(\pi) + C_i(4\pi) - 2C_i(2\pi)]\}$$

$$= 60\{0.5772 + 1.8378 - (-0.0227) + \tfrac{1}{2}(1)[0.5772 + 1.1447 - 0.006 - 2(-0.0227)]\}$$

$$R_r = 199.099 \Rightarrow R_{in} = \frac{199.099}{\sin^2(\pi)} = \infty$$

$$\Gamma = \frac{\infty - 50}{\infty + 50} = \frac{1 - 50/\infty}{1 + 50/\infty} = 1 \Rightarrow VSWR = \infty$$

4-20. $R_r = 80\pi^2\left(\frac{\ell}{\lambda}\right)^2$, $a = 10^{-4}\lambda$, $f = 10\,MHz$, $\sigma = 5.7 \times 10^7\,S/m$

$$R_L = R_{hf} = \frac{\ell}{P}\sqrt{\frac{\omega\mu}{2\sigma}} = \frac{\ell}{C}\sqrt{\frac{\omega\mu_0}{2\sigma}} = \frac{\ell}{2\pi a}\sqrt{\frac{\omega\mu_0}{2\sigma}} = \frac{\ell}{2\pi \times 10^{-4}\lambda}\sqrt{\frac{2\pi \times 10^7 (4\pi \times 10^{-7})}{2\cdot(5.7\times 10^7)}}$$

$$R_L = R_{hf} = 1.3245\left(\frac{\ell}{\lambda}\right) \qquad e_{cd} = \frac{R_r}{R_L + R_r}$$

(a) $\ell = \lambda/50$ ; $R_r = 80\pi^2\left(\frac{\lambda}{50\lambda}\right)^2 = 0.316$ ohms

$$R_L = R_{hf} = 1.3245\left(\frac{1}{50}\right) = 0.02649$$

$$e_{cd} = \frac{R_r}{R_L + R_r} \times 100 = \frac{0.316 \times 100}{0.02649 + 0.316} = 92.26\,\%$$

(b) $\ell = \lambda/4$; From Prob. 4.19 $R_r = 6.8388$

$$R_L = R_{hf} = \frac{1.3245}{4} = 0.3311$$

$$e_{cd} = \frac{6.8388 \times 100}{6.8388 + 0.3311} = 95.38\%$$

(Continued)

4-20. (Cont'd)

(c) $\ell = \lambda/2$; From Prob. 4-19, $R_r = 73.13$

$$R_L = R_{hf} = \frac{1.3245}{2} = 0.66225$$

$$e_{cd} = \frac{73.13 \times 100}{73.13 + 0.66225} = 99.10\%$$

(d). $\ell = \lambda$; From Prob 4-19, $R_r = 199.099$

$$R_L = R_{hf} = 1.3245$$

$$e_{cd} = \frac{199.099}{199.099 + 1.3245} \times 100 = 99.34\ \%$$

4-21. $$H_\theta = j\frac{k e^{-jkr}}{4\pi r \cdot \eta} \cdot \sin\theta \left[\int_{-\ell/2}^{\ell/2} I_m \cdot \cos\left(\frac{\pi}{\ell} z'\right) e^{jkz'\cos\theta}\, dz'\right]$$

$$H_\theta = j\frac{k \cdot I_m e^{jkr}}{4\pi r \cdot \eta} \cdot \sin\theta \int_{-\ell/2}^{\ell/2} \cos\left(\frac{\pi}{\ell} z'\right) e^{jkz'\cos\theta}\, dz'$$

Using the same formula in Problem 4-18 (b).

$$H_\theta = j\ \frac{I_m \cdot k \cdot e^{jkr}}{\eta\, 4\pi r} \cdot \sin\theta \cdot \frac{\pi}{\ell} \cdot \frac{2\cos\left(\frac{k\ell}{2}\cos\theta\right)}{\left(\frac{\pi}{\ell}\right)^2 - k^2\cos^2\theta} = j\frac{I_m k}{\eta 4\pi r} e^{jkr} \cdot \frac{1}{k} \cdot \frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

$$= j\frac{I_m e^{-jkr}}{\eta\, 2\pi r} \cdot \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

$$E_\phi = -\eta H_\theta = -j\frac{I_m e^{-jkr}}{2\pi r} \cdot \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

4-22. (a) $VSWR = \dfrac{1+|\Gamma|}{1-|\Gamma|} \Rightarrow |\Gamma| = \dfrac{VSWR-1}{VSWR+1} = \left|\dfrac{2-1}{2+1}\right| = \left|\dfrac{1}{3}\right|$

$$|\Gamma| = \left|\frac{1}{3}\right| = \left|\frac{z_{in}-z_c}{z_{in}+z_c}\right| = \left|\frac{z_{in}/z_c - 1}{z_{in}/z_c + 1}\right| = \begin{cases} \left|\dfrac{2-1}{2+1}\right| \Rightarrow \dfrac{z_{in}}{z_c} = 2 \\ \left|\dfrac{\frac{1}{2}-1}{\frac{1}{2}+1}\right| \Rightarrow \dfrac{z_{in}}{z_c} = \dfrac{1}{2} \end{cases}$$

Largest $\dfrac{z_{in}}{z_c} = 2 \Rightarrow z_{in} = 2z_c = 100$

(b) $R_{in} = 11.14\, G^{4.17}$ $\qquad \lambda/2 < \ell < 2\lambda/\pi$

$\qquad\qquad\qquad\qquad\quad \pi/2 < k\ell/2 < 2$

$100 = 11.14\, G^{4.17}$

$\dfrac{100}{11.14} = G^{4.17}$, $\quad 8.9767 = G^{4.17}$

$\log_{10}(8.9767) = 4.17 \log_{10}(G)$, $\quad 0.953 = 4.17 \log_{10} G$

$0.2286 = \log_{10} G$, $\quad G = 10^{0.2286} = 1.6928 = \dfrac{k\ell}{2} = 96.99°$

$k\ell = 2(1.6928)$, $\quad \ell = \dfrac{2(1.6928)\lambda}{2\pi} = \dfrac{1.6928}{\pi}\lambda = 0.5388\lambda$

$\ell = 0.5388\lambda$

(c) $R_{in} = \dfrac{R_r}{\sin^2(\frac{k\ell}{2})} \Rightarrow R_r = R_{in}\sin^2(\frac{k\ell}{2}) = 100\sin^2(96.99°)$

$R_{in} = 100(0.9926)^2 = 100(0.9852) = 98.52$ ohms

$R_{in} = 98.52$ ohms

4-23. $W_{av} = \dfrac{1}{2\eta}(|E_\theta|^2 + |E_\phi|^2) = \dfrac{1}{2\eta}\left[\dfrac{\omega^2\mu^2\sin^2\theta}{16\pi^2 r^2} I_0^2 (k^2A_1^2 + 4A_2^2)\right]$

$$P_{rad} = \frac{1}{2\eta}\frac{\omega^2\mu^2 I_0^2}{16\pi^2}\left[\int_0^{2\pi}\int_0^{\pi} \sin^3\theta \, d\theta \, d\phi\right]\left[k^2A_1^2 + 4A_2^2\right]$$

$$P_{rad} = \frac{\omega^2\mu^2 I_0^2 (k^2A_1^2 + 4A_2^2)}{12\pi\eta}, \qquad \left(\leftarrow \int_0^{2\pi}\int_0^{\pi} \sin^3\theta \, d\theta \, d\phi = \frac{8\pi}{3}\right)$$

$$\Rightarrow R_{rad} = \frac{2P_{rad}}{I_0^2} = \frac{\omega^2\mu^2 (k^2A_1^2 + 4A_2^2)}{6\pi\eta}$$

Elliptical polarization since

$$\vec{E}(t) = \frac{-\omega\mu k\sin\theta}{4\pi r} I_0 A_1 \sin(\omega t - kr)\,\hat{a}_\theta + \frac{\omega\mu k\sin\theta}{2\pi r} I_0 A_2 \cos(\omega t - kr)\,\hat{a}_\phi$$

4-24. Dipole with $\ell = \lambda/2$

$$\underline{E}^a \simeq \hat{a}_\theta\, j\eta \frac{I_0 e^{-jkr}}{2\pi r}\left\{\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right\}$$

$$\simeq -\hat{a}_\theta\, j\eta \frac{I_0 k e^{-jkr}}{2\pi r}\left\{\frac{-\cos(\frac{\pi}{2}\cos\theta)}{k\sin\theta}\right\}$$

$$\simeq -\hat{a}_\theta\, j\eta \frac{I_0 k e^{-jkr}}{2\pi r}\left\{-\frac{\lambda}{2\pi}\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right\}$$

$$\underline{E}^a \simeq -j\eta \frac{I_0 k e^{jkr}}{4\pi r}\left\{-\hat{a}_\theta \frac{\lambda}{\pi}\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right\}$$

(a) $\underline{h}_e(\theta) = -\hat{a}_\theta \frac{\lambda}{\pi}\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}$

(b) $|\underline{h}_e(\theta)|_{max} = \left|-\hat{a}_\theta \frac{\lambda}{\pi}\cdot\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right|_{max}^{\theta=90^\circ} = \frac{\lambda}{\pi} = 0.3183\lambda$

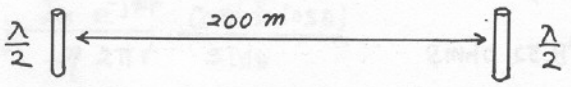
(c) $\frac{|h_e(\theta)|_{max}}{\ell=\lambda/2} = \frac{\lambda/\pi}{\lambda/2} = \frac{2}{\pi} = 0.6366$

which is 63.66 % of $\ell$

(d) $V_{oc} = |\underline{h}_e \cdot \underline{E}^i|_{\theta=90^\circ} = \left|-\hat{a}_\theta \frac{\lambda}{\pi}\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\cdot -\hat{a}_\theta \frac{10^{-3}}{\lambda}\,V\right|_{\theta=90^\circ}$

$= \frac{\lambda}{\pi}\left(\frac{10^{-3}}{\lambda}\right) = \frac{10^{-3}}{\pi} = 3.183\times10^{-4}$ Volts

4-25.




$\theta = 90^\circ, \phi = 40^\circ$

At $f = 300$ MHz, $\lambda = \frac{c}{f} = 1$ m

$\Rightarrow \frac{2D^2}{\lambda} = \frac{2(\frac{\lambda}{2})^2}{\lambda} = 0.5$ m

$r = 200\text{m} \gg 0.5\text{m}$

$$P_r = \left(\frac{\lambda}{4\pi r}\right)^2 G_{Tot}\, G_{Or} = \left(\frac{\lambda}{4\pi r}\right)^2 D_{ot}\, D_{or}$$

← for lossless antenna

Now since $D_{ot} = D_{or} = 1.643$ for $\frac{\lambda}{2}$ dipole.

$$P_r = \left(\frac{1}{4\pi\cdot 200}\right)^2 (1.643)(1.643) \quad W = 0.2\,mW$$

4-26. The time average power density ($W_{av} = \frac{1}{2}\frac{|E|^2}{\eta}$)

$$W_{av} = \eta\frac{|I_0|^2}{8\pi^2 r^2}\left[\frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}\right], \quad P_{rad} = \eta\frac{|I_0|^2}{4\pi}\int_0^{\pi}\frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}\,d\theta$$

$$P_{rad} = \frac{1}{2}R_{rad}|I_0|^2, \quad R_{rad} = \frac{\eta}{4\pi}[\gamma + \ln(2\pi) - Ci(2\pi)] = 30[0.57712 + 1.838 + 0.02]$$

$$R_{rad} = 73.0523.$$

$$P_{rad} = (0.5\cdot 100) = 50 \text{ watts} \cdot \quad 50 = \frac{1}{2}(73.0523)|I_0|^2.$$

$$|I_0|^2 = 1.36888.$$

At $r = 500\text{m}$, $\theta = 60°$, $\phi = 0°$

$$W_{av} = 120\pi\cdot\frac{1.36888}{8\pi^2(500)^2}\cdot\left[\frac{\cos^2(\frac{\pi}{2}\cos 60°)}{\sin^2 60°}\right]$$

$$= 15\cdot\frac{1.36888}{\pi\cdot 25\times 10^4}\cdot(0.6667)$$

$$= 1.743\times 10^{-5} \text{ watts/m}^2.$$

4-27. (a) $I_{in} = I_0 \sin[k(\frac{\ell}{2} + |z|)]$

$\ell = \lambda/4$, $z = \lambda/8$

$$I_{in} = I_0\cdot\sin[k(\frac{\lambda}{4} \pm \frac{\lambda}{8})] = I_0\sin[k\frac{\lambda}{8}] = I_0\sin[\frac{2\pi}{\lambda}\frac{\lambda}{8}]$$

$$= I_0\sin(\frac{\pi}{4})$$

$$I_{in} = 0.707\, I_0$$

$$R_{in} = (\frac{I_0}{I_{in}})^2 R_r = (\frac{I_0}{.707 I_0})^2 R_r = 2R_r = 2(73) = 146$$

$$X_{in} = (\frac{I_0}{I_{in}})^2 X_m = (\frac{I_0}{.707 I_0})^2 X_m = 2X_m = 2(42.5) = 85$$

$$Z_{in} = R_{in} + jX_{in} = 146 + j85$$

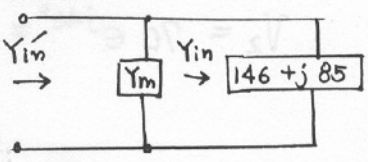
(b) $$Y_{in} = \frac{1}{146+j85}\cdot\frac{146-j85}{146-j85} = \frac{146-j85}{168.941} = (5.115 - j2.978)\times 10^{-3}$$

$$Y_m = +j\,2.978\times 10^{-3} \Rightarrow X_m = \frac{1}{Y_m} = -j\,335.776 \quad \text{(Capacitive)}$$

(c) $Y_{in}' = 5.115\times 10^{-3} \Rightarrow Z_{in} = \frac{}{5.115\times 10^{-3}} = 195.503$ ohms

$$|\Gamma| = \left|\frac{195.503 - 300}{195.503 + 300}\right| = \frac{104.4966}{495.503} = 0.21$$

$$VSWR = (1 + 0.211)/(1 - 0.211) = 1.5346$$



4-28. $\ell = \lambda/2$, $Z_c = 50$ ohms

$Z_{in} = 73 + j42.5$, $Y_{in} = \frac{1}{Z_{in}} = \frac{1}{73+j42.5} \cdot \frac{73-j42.5}{73-j42.5}$

$Y_{in} = 0.01023 - j0.0059563 = (10.23 - j5.9563)\times 10^{-3} = G_{in} - jB_{in}$

$B_{in} = \omega C_{in} = 2\pi f C_{in} \Rightarrow C_{in} = \frac{B_{in}}{2\pi f} = \frac{5.9563\times 10^{-3}}{2\pi\cdot(10\times 10^8)} = 0.94797\times 10^{-12}$

$\therefore C_{in} = 0.94797$ pF

$G_{in} = 10.23\times 10^{-3}$

$R_{in} = \frac{1}{G_{in}} = 97.75$, $\Gamma_{in} = \frac{R_{in} - Z_c}{R_{in} + Z_c} = \frac{97.75 - 50}{97.75 + 50} = 0.3232$

$VSWR = \frac{1 + |\Gamma_{in}|}{1 - |\Gamma_{in}|} = \frac{1 + 0.3232}{1 - 0.3232} = 1.955$

4-29. $\underline{E} = -\hat{a}_\theta\, j\frac{\omega\mu b I_0 e^{-jkr}}{4\pi r}\cdot \left.\frac{\sin(\frac{kb}{2}\cos\theta)}{\frac{kb}{2}\cos\theta}\right|_{\theta=90^\circ} = -\hat{a}_\theta \cdot j\cdot\frac{\omega\mu b I_0 e^{-jkr}}{4\pi r}\cdot(1)$

$\underline{E}|_{\theta=90^\circ} = -\hat{a}_\theta\cdot j\cdot\frac{\omega\mu I_0 e^{-jkr}}{4\pi r}\, b = -j\,\frac{\omega\mu I_0 e^{-jkr}}{4\pi r}\cdot h_e(\theta)$

$h_e(\theta) = \hat{a}_\theta \cdot b$

$\underline{E}^{inc}\Big|_{\theta=90^\circ} = \hat{a}_\theta\, j\eta\,\frac{kI_0\ell e^{-jkr}}{4\pi r}\sin\theta\Big|_{\theta=90^\circ} = \hat{a}_\theta\cdot j\eta\cdot\frac{kI_0\ell e^{-jkr}}{4\pi r}$

$P = \frac{|\underline{h}_e(\theta)\cdot\underline{E}^{inc}|^2}{|h_e(\theta)|^2\cdot|\underline{E}^{inc}|^2} = \frac{|\eta\frac{bkI_0\ell}{4\pi r}|^2}{|b|^2\cdot|\eta\frac{kI_0\ell}{4\pi r}|^2} = 1.$

$P(dB) = 10\log_{10}(1) = 0$ dB

4-30. $V_1 = 4e^{j20^\circ} = C\,[\hat{a}_y]\cdot\left[\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right]$, at $z = 0$

$V_1 = 4e^{j20^\circ} = j\cdot C\cdot\frac{1}{\sqrt{2}} \Rightarrow C = -j4\sqrt{2}\,e^{j20^\circ}$, $C = 4\sqrt{2}\,e^{-j70^\circ}$

$V_2 = (4\sqrt{2}\,e^{-j70^\circ})[10(2\hat{a}_x + \hat{a}_y e^{j30^\circ})]\cdot\left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right) = 40\sqrt{2}\,e^{-j70^\circ}\left[\frac{2 + j e^{j30^\circ}}{\sqrt{2}}\right]$

$= 40\,e^{-j70^\circ}[2 + j(\cos 30^\circ + j\sin 30^\circ)]$

$= 40\,e^{-j70^\circ}[1.5 + j0.866] = 40\,e^{-j70^\circ}[1.73\,e^{j30^\circ}]$

$V_2 = 70\,e^{-j40^\circ} = 53.6 - j45^\circ$

4-31. $l = 3\,cm$, $\lambda = 5\,cm$, $I = 10 e^{j60^\circ}$

$r > \frac{2D^2}{\lambda} = \frac{2\times 3^2}{5} = \frac{18}{5} = 3.6\,cm \Rightarrow 10\,cm$ is in the far field.

$\frac{l}{\lambda} = \frac{3}{5} = 0.6 \Rightarrow$ length of dipole is finite, $\frac{kl}{2} = \pi \cdot \frac{l}{\lambda} = 0.6\pi$

$$E_\theta \simeq j\eta \frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta}\right] = j\eta \frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos(0.6\pi\cos\theta) + 0.309}{\sin\theta}\right]$$

$H_\phi \simeq \frac{E_\theta}{\eta}$, $\left(\leftarrow \left.\frac{\cos(0.6\pi\cdot\cos 45^\circ) + 0.309}{\sin 45^\circ}\right|_{\theta=45^\circ} = 0.7703\right)$

$e^{-jkr} \Rightarrow kr = \frac{2\pi}{\lambda} r = \frac{2\pi}{5}\cdot 10 = 4\pi = 12.5663$ rad

$$\Rightarrow E_\theta = j120\pi \cdot \frac{I_0 e^{j60^\circ} \cdot e^{-j4\pi}}{2\pi(0.1\,m)} \cdot (0.7703) = 4620\, e^{j11.52}$$

$|E_\theta| = 4620$ V/m, $|H_\phi| = \frac{4620}{120\pi} = 12.25$ Ampere

4-32. Using equation (4-79)

$$R_{in} = \frac{R_r}{\sin^2\left(\frac{kl}{2}\right)} \simeq \frac{120 \text{ ohms}}{\sin^2(0.6\pi)} = 132.668 \text{ ohms}$$

4-33. $\frac{kl}{2} = \frac{3\pi}{4}$, $kl = \frac{3\pi}{2}$, $2kl = 3\pi$

a) Using (8-60a), (8-60b)

$R_r = 185.808$, $X_r = 190.7967$

b). Using (8-61a), (8-61b)

$$R_{in} = \frac{185.808}{\sin^2(3/4\,\pi)} = 371.617, \quad X_{in} = \frac{190.7967}{\sin^2(3/4\cdot\pi)} = 385.5936$$

c) $\Gamma = \frac{371.617 - 300}{371.617 + 300} = 0.10663$,

$$VSWR = \frac{1 + 0.10663}{1 - 0.10663} = 1.2387$$

4-34. $\ell = 0.625\lambda$,

a. Using (8-60a), (8-60b)

$R_r = 131.9415$, $X_r = 146.131638$

b. Using (8-61a), (8-61b)

$R_{in} = 154.579$, $X_{in} = 171.203$

c. $\Gamma = \dfrac{154.579 - 300}{154.579 + 300} = -0.3199$, $VSWR = \dfrac{1+|0.3199|}{1-|-0.3199|} = 1.9407$

4-35. a. $\ell = 200$ m, $a = 1$ m, $f = 150$ kHz $\rightarrow \lambda = 2000$ meters

Using (11-37).

$$Z_{in} \simeq 20\pi^2\left(\frac{\ell}{\lambda}\right)^2 - j120\,\frac{\left[\ln\left(\frac{\ell}{2a}\right) - 1\right]}{\tan\left(\pi\frac{\ell}{\lambda}\right)} \simeq 20\pi^2\left(\frac{1}{10}\right)^2 - j120\cdot\frac{[\ln(100) - 1]}{\tan(\pi/10)}$$

$Z_{input} = 2 + Z_{in} = 2 + 1.9739 + j\,1377.07$

$Z_{input} = 3.9739 + j\,1377.07$

b. Radiation efficiency $= 100\cdot\dfrac{R_r}{R_L + R_r} = 100\cdot\dfrac{1.9739}{3.9739} = 49.67\%$

c. $RPF = \dfrac{R_r}{|Im(Z_{input})|} = \dfrac{1.9739}{1.377\times10^{+3}} = 1.4335\times10^{-3}$

d. $X = -Im(Z_{input}) = -1377.07$

$n = \sqrt{\dfrac{R_r + R_L}{Z_0}} = \sqrt{\dfrac{3.9739}{50}} = 0.282$

e The answer to this part was found by manually entering values of x until $|\Gamma| = 0.333$ was obtained.

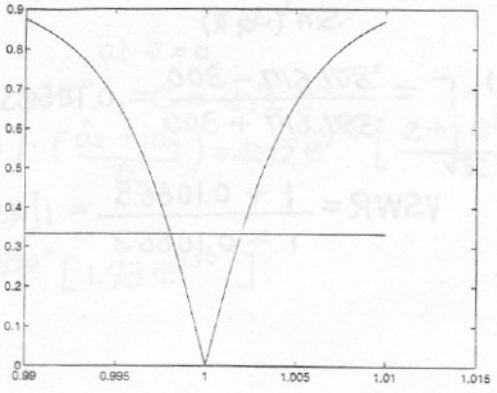
The values obtained are

$x_1 = 0.99803$

$x_2 = 1.00198$

The corresponding percent bandwidth is

$BW = (x_2 - x_1)\times100\% = 0.395\%$



4-36. $E_\theta = j\eta \frac{kI_0 l e^{-jkr}}{2\pi r} \sin\theta \cdot \cos(kh\cos\theta)$ ; $0 \le \theta \le \pi/2$, $0 \le \phi \le 2\pi$

$$\underline{W}_{ave} = \frac{1}{2}\mathrm{Re}[\underline{E} \times \underline{H}^*] = \frac{\hat{a}_r}{2\eta}|E_\theta|^2 = \hat{a}_r \frac{\eta}{2}\left|\frac{kI_0 l}{2\pi r}\right|^2 \sin^2\theta \cdot \cos^2(kh\cos\theta)$$

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi/2} \underline{W}_{ave} \cdot \hat{a}_r r^2 \sin\theta \, d\theta \, d\phi = \frac{\eta}{2}\left|\frac{kI_0 l}{2\pi}\right|^2 \int_0^{2\pi}\int_0^{\pi/2} \sin^3\theta \cos^2(kh\cos\theta)\, d\theta\, d\phi$$

$$= \frac{\eta}{\pi}\left|\frac{kI_0 l}{2}\right|^2 \cdot \int_0^{\pi/2} \sin^3\theta \cos^2(kh\cos\theta) d\theta = \frac{\eta}{\pi}\left|\frac{kI_0 l}{2}\right|^2 \int_0^{\pi/2} \sin^3\theta \left[\frac{1+\cos(2kh\cos\theta)}{2}\right] d\theta$$

$$= \frac{\eta}{2\pi}\left|\frac{kI_0 l}{2}\right|^2 \left\{ \int_0^{\pi/2} \sin^3\theta \, d\theta + \int_0^{\pi/2} \sin^3\theta \cdot \cos(2kh\cos\theta) d\theta \right\}$$

$$P_{rad} = \frac{\eta}{2\pi}\left|\frac{kI_0 l}{2}\right|^2 \{ I_1 + I_2 \}$$

where $I_1 = \int_0^{\pi/2} \sin^3\theta \, d\theta = -\frac{1}{3}\cos\theta\,(\sin^2\theta + 2)\Big|_0^{\pi/2} = \frac{2}{3}$

$$I_2 = \int_0^{\pi/2} \sin^3\theta \cos(kh\cos\theta) d\theta = \int_0^{\pi/2} \sin^2\theta \cdot \cos(kh\cos\theta) \sin\theta \, d\theta$$

Let $u = \sin^2\theta$ $\qquad v = -\frac{1}{2kh}\sin(2kh\cos\theta)$

$du = 2\sin\theta\cos\theta\, d\theta$ $\qquad dv = -\frac{\cos(2kh\cos\theta)}{2kh} \cdot d(2kh\cos\theta)$

Thus

$$I_2 = -\frac{\sin^2\theta}{2kh} \cdot \sin(2kh\cos\theta)\Big|_0^{\pi/2} + \frac{2}{2kh}\int_0^{\pi/2} \cos\theta \cdot \sin(2kh\cos\theta) \sin\theta \, d\theta$$

Let $u = \cos\theta$ $\qquad dv = -\frac{1}{2kh}\sin(2kh\cos\theta) d(2kh\cos\theta)$

$du = -\sin\theta\, d\theta$ $\qquad v = \frac{1}{2kh}\cos(2kh\cos\theta)$

$$I_2 = 0 + \frac{2}{2kh}\left\{ \frac{\cos\theta}{2kh}\cos(2kh\cos\theta)\Big|_0^{\pi/2} + \frac{1}{2kh}\int_0^{\pi/2} \cos(2kh\cos\theta)\sin\theta\, d\theta \right\}$$

$$= \frac{2}{2kh}\left\{ -\frac{1}{2kh}\cos(2kh) - \frac{1}{(2kh)^2}\sin(2kh\cos\theta)\Big|_0^{\pi/2} \right\} = 2\left\{ -\frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3} \right\}$$

Therefore

$$P_{rad} = \frac{\eta}{2\pi}\left|\frac{kI_0 l}{2}\right|^2 \cdot \{I_1 + I_2\} = \pi\eta\left|\frac{I_0 l}{\lambda}\right|^2 \cdot \left[\frac{1}{3} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3}\right]$$

4-37. $E_\theta = C_1 \sin\theta \cos(kh\cos\theta)$, where $C_1 = j\eta \frac{kI_0 \ell e^{-jkr}}{2\pi r}$

a. $E_\theta\big|_{\theta=30^\circ} = C_1 \sin\theta\cdot\cos(kh\cos\theta)\big|_{\theta=30^\circ} = 0 \Rightarrow \cos(kh\cos\theta)\big|_{\theta=30^\circ} = 0$

$kh\cos(30^\circ) = \frac{2\pi}{\lambda} h\,(0.867) = \cos^{-1}(0) = \frac{\pi}{2} \Rightarrow h = \frac{1}{4(0.867)}\cdot\lambda = 0.288\lambda$

b. $D_o = \dfrac{2}{\left[\frac{1}{3} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3}\right]}$, $2kh = 2\cdot\left(\frac{2\pi}{\lambda}\right)\cdot(0.288\lambda) = 3.632$

$D_o = \dfrac{2}{\left[\frac{1}{3} - \frac{\cos(3.632)}{(3.632)^2} + \frac{\sin(3.632)}{(3.632)^3}\right]} = \dfrac{2}{[\frac{1}{3} + 0.06689 - 0.00983]} = 5.12 = 7.1\,\text{dB}$

c. $R_r = 2\pi\eta\left(\frac{\ell}{\lambda}\right)^2\left[\frac{1}{3} - \frac{\cos(3.632)}{(3.632)^2} + \frac{\sin(3.632)}{(3.632)^3}\right] = 2\pi(377)\left(\frac{1}{50}\right)^2[0.39] = 0.37\,\text{ohm}$

4-38. $E_\theta = C_1 \sin\theta\cdot\cos(kh\cos\theta)$, where $C_1 = j\eta \frac{kI_0 \ell e^{-jkr}}{2\pi r}$

$E_\theta\big|_{h=2\lambda} = C_1\cdot\sin\theta_n\cdot\cos(kh\cos\theta_n)\big|_{h=2\lambda} = 0 \Rightarrow \sin\theta_n = 0,\ \cos(kh\cos\theta_n)\big|_{h=2\lambda} = 0$

$\sin\theta_n = 0 \Rightarrow \theta_n = 0^\circ$

$\cos(kh\cos\theta_n)\big|_{h=2\lambda} = \cos(4\pi\cos\theta_n) = 0 \Rightarrow 4\pi\cos\theta_n = \cos^{-1}(0) = \pm\left(\frac{2n+1}{2}\right)\pi,\ n = 0, 1, 2, \cdots$

$\theta_n = \cos^{-1}[\pm(2n+1)/8]$, $n = 0, 1, 2, 3, 4, \cdots$

$n=0$: $\theta_0 = \cos^{-1}(\pm\frac{1}{8}) = 82.82^\circ$

$n=1$; $\theta_1 = \cos^{-1}(\pm\frac{3}{8}) = 67.98^\circ$

$n=2$; $\theta_2 = \cos^{-1}(\pm\frac{5}{8}) = 51.32^\circ$

$n=3$; $\theta_3 = \cos^{-1}(\pm\frac{7}{8}) = 28.96^\circ$

} for $0^\circ \le \theta \le 90^\circ$ (for $90^\circ \le \theta \le 180^\circ$, the field is zero)

$n=4$; $\theta_4 = \cos^{-1}(\pm\frac{9}{8})$ = Does not exist. The same holds for $n \ge 5$.

Therefore where the field vanishes for $0^\circ \le \theta \le 90^\circ$, are

$\theta = 0^\circ$, $28.96^\circ$, $51.32^\circ$, $67.98^\circ$, and $82.82^\circ$

4-39. $E_\theta = C_1\cdot\sin\theta\cdot\cos(kh\cos\theta)$ where $C_1 = j\eta\frac{kI_0 e^{jkr}\ell}{2\pi r}$

$E_\theta|_{\theta=60^\circ} = C_1\cdot\sin(60^\circ)\cdot\cos(kh_n\cos(60^\circ)) = 0 \Rightarrow \cos(kh_n\cos(60^\circ)) = 0$

$kh_n\cos(60^\circ) = kh_n(\frac{1}{2}) = \frac{\pi}{\lambda}h_n = \cos^{-1}(0) = \pm\left(\frac{2n+1}{2}\right)\pi,\quad n = 0, 1, 2, 3, \cdots$

Choosing the positive values

$h_n = \left(\frac{2n+1}{2}\right)\lambda$, $n = 0, 1, 2, 3, \cdots$

$h_n = 0.5\lambda, 1.5\lambda, 2.5\lambda, 3.5\lambda, 4.5\lambda$

4.40. $E_\theta$ (4-99) $\simeq C\cdot\sin\theta\cdot[2\cos(kh\cos\theta)] \Rightarrow AF = 2[\cos(kh\cos\theta)]_{max} = \pm 2$

$\cos(kh\cos\theta_m) = \pm 1$

a. $kh\cos\theta_m = \cos^{-1}(\pm 1) \Rightarrow \theta_m = \frac{1}{kh}\cos^{-1}(\pm 1) = \pm\frac{m\pi}{kh} = \frac{\pm m\pi}{\frac{2\pi}{\lambda}(\frac{3\lambda}{2})} = \pm\frac{m}{3}$, $m = 0,1,2,\cdots$

$\theta_m = \cos^{-1}(\pm m/3)$ ; $m = 0,1,2,3,\cdots$

$m=0$ ; $\theta_0 = \cos^{-1}(\pm 0) = 90°$

$m=1$ ; $\theta_1 = \cos^{-1}(\pm 1/3) = \begin{cases} \cos^{-1}(1/3) = 70.5288° \\ \cos^{-1}(-1/3) = 109.4712° \ (\Rightarrow \text{Below Ground Plane}) \end{cases}$

$m=2$ ; $\theta_2 = \cos^{-1}(\pm 2/3) = \begin{cases} \cos^{-1}(2/3) = 48.1897° \\ \cos^{-1}(-2/3) = 131.8103° \ (\Rightarrow \text{Below Ground Plane}) \end{cases}$

$m=3$ : $\theta_3 = \cos^{-1}(\pm 1) = \begin{cases} \cos^{-1}(1) = 0° \\ \cos(-1) = 180° \ (\Rightarrow \text{Below Ground Plane}) \end{cases}$

$m=4$ : $\theta_4 = \cos^{-1}(\pm 4/3) \Rightarrow$ does not exist

b. $E_{\theta m} = C\cdot\sin\theta\,[2\cdot\cos(kh\cos\theta)]_{max} = \pm 2C$ , where $\theta = 90°$

c. $\frac{E_\theta}{E_{\theta m}} = \sin\theta\cdot\cos(kh\cos\theta)$

$\theta = 0°$ : $\frac{E_\theta}{E_{\theta m}} = 0 \Rightarrow \frac{E_\theta}{E_{\theta m}} = 20\log_{10}(0) = -\infty$ dB

$\theta = 48.1897°$ : $\frac{E_\theta}{E_{\theta m}} = \sin\theta\cdot\cos(kh\cos\theta)\Big|_{h=\frac{3\lambda}{2}} = 0.7454 \Rightarrow \frac{E_\theta}{E_{\theta m}} = 20\log_{10}(0.7454)$

$\frac{E_\theta}{E_{\theta m}} = -2.55$dB

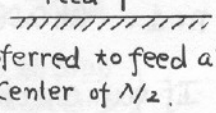
$\theta = 70.5288°$ : $E_\theta / E_{\theta m} = \sin\theta\cdot\cos(kh\cos\theta)\Big|_{h=\frac{3\lambda}{2}} = 0.9428 \Rightarrow \frac{E_\theta}{E_{\theta m}} = 20\log_{10}(0.9428)$

$\frac{E_\theta}{E_{\theta m}} = -0.5115$dB

$\theta = 90°$ : $E_\theta/E_{\theta m} = \sin\theta\cdot\cos(kh\cos\theta)\Big|_{h=\frac{3\lambda}{2}} = 1 \Rightarrow \frac{E_\theta}{E_{\theta m}} = 20\log_{10}(1) = 0$ dB

4.41. a. $Z_{im}(\ell=\lambda/2)\Big| = \frac{1}{2}Z_{im}(\ell=\lambda)\Big| \simeq \frac{1}{2}(R_{im}+jX_{im})\Big|_{\ell=\lambda}$

above ground plane  free space




Referred to feed at center of $\lambda/2$. From Fig 4-19.

From Problem 4.19 $\Rightarrow R_{im} = R_r = 199.099$

From Figure 4.19 $\Rightarrow X_{im}(\ell=\frac{\lambda}{2})\Big| \simeq 62.5$

above ground plane

Therefore

$Z_{im}(\ell=\lambda/2)\Big| = \frac{199.099}{2} + j62.5 = 99.5495$

above ground plane

(Continued)

4-41 (Cont'd)

b. $Z_{in} = \dfrac{Z_{im}}{\sin^2(\frac{kl}{2})} = \dfrac{99.5495 + j62.5}{\sin^2(\pi)} = \infty$

c. $\Gamma = \dfrac{Z_{in} - Z_c}{Z_{in} + Z_c} = \dfrac{\infty - 50}{\infty + 50} = \dfrac{1 - 50/\infty}{1 + 50/\infty} = 1$

$VSWR = \dfrac{1+|\Gamma|}{1-|\Gamma|} = \dfrac{1+1}{1-1} = \infty$

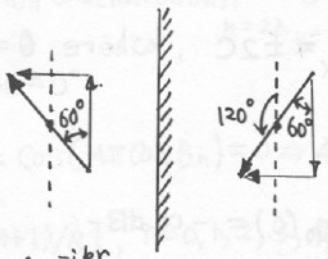
. $X_{im}$ can also be obtained using (8-60b). For $l = \lambda \Rightarrow kl = 2\pi$, $2kl = 4\pi$, Thus

$$X_{im}(l=\lambda/2)\Big|_{\text{above ground plane}} = \frac{1}{2} X_{im}(l=\lambda)\Big|_{\text{free space}} = \frac{\eta}{8\pi}\left\{2S_i(kl) + \cos(kl)\left[2S_i(kl) - S_i(2kl)\right]\right\}$$

$$= \frac{120\pi}{80\pi}\left\{2S_i(2\pi) + \cos(2\pi)\left[2S_i(2\pi) - S_i(4\pi)\right]\right\} = 15\left\{2(1.418) + \left[2(1.418) - 1.492\right]\right\}$$

$= 62.7$

4-42.




4-43. $E_\theta \simeq j\eta \dfrac{kI_0 l e^{-jkr}}{4\pi r} \sin\theta \cdot [2\cos(kh\cos\theta)]$

$|A_F|_{max} = |\cos(kh\cos\theta)|_{max} = 1$ when $kh\cos\theta_{max} = \pi$

$kh\cos\theta_{max} = \pi$, $kh\cos(60°) = \pi$,

$\dfrac{2\pi}{\lambda} \cdot h \cdot (\frac{1}{2}) = \pi$, $h = \lambda$

No matter what the height is when $\theta = 90°$, it is a maximum.

So you always have a maximum at $\theta = 90°$. If you want a maximum at $\theta = 60°$, then $kh\cos\theta = n\pi$, $(n = 1, 2, 3, \cdots)$ leads to a maximum at $\theta = 60°$.

$n = 1$: $kh\cos\theta|_{max} = \pi$, $h = \lambda$ leads to maxima at $\theta = 90°, 60°$

If you check closely, it also leads to a maximum at $\theta = 0°$.

So you can not only have one maximum at $\theta = 60°$

4-44. $E_\theta \sim C_1 \cdot \sin\theta \cdot \cos(kh\cos\theta)\big|_{\theta=80^\circ} = 0$

$\cos(kh\cos\theta)\big|_{\theta=80^\circ} = 0, \quad kh\cos\theta\big|_{\theta=80^\circ} = \frac{\pi}{2}, \quad \frac{2\pi}{\lambda} h\cos\theta\big|_{\theta=80^\circ} = \frac{\pi}{2}$

$h = \frac{\lambda}{4\cos\theta}\Big|_{\theta=80^\circ} = \frac{\lambda}{4(0.1736)} = \frac{\lambda}{0.6946} = 1.4397\cdot\lambda$

$h = 1.4397\lambda, \quad \lambda = \frac{3\times10^8}{50\times10^6} = \frac{30\times10^7}{5\times10^7} = 6 \text{ meters}$

$h = 1.4397\cdot\lambda = 1.4397\cdot(6) = 8.6382 \text{ m}$

$h = 8.6382$ meters

4-45. Using the coordinate system of Figure 4.21 the total field is given by (4-116) or

$$E_\psi = j\eta \frac{kI_0 l e^{-jkr}}{4\pi r}\sqrt{1-\sin^2\theta\sin^2\phi}\,[2j\sin(kh\cos\theta)], \quad 0\le\theta\le\pi, \; 0\le\phi\le2\pi$$

However if we rotate the axes so that the z axis is parallel to the axis of the element and y is vertical to the ground, the total E-field can be written as

$$E_\theta = j\eta \frac{kI_0 e^{-jkr}}{4\pi r}\cdot\sin\theta\cdot[2j\sin(kh\sin\theta\cdot\sin\phi)], \quad \text{and}$$

$$P_{rad} = \int_0^\pi\int_0^\pi \underline{W}_{ave}\cdot\hat{a}_r r^2\sin^2\theta\, d\theta d\phi = \frac{1}{2\eta}\int_0^\pi\int_0^\pi |E_\theta|^2 r^2\sin\theta\, d\theta d\phi$$

$$P_{rad} = \frac{\eta}{2}\left|\frac{kI_0 l}{2\pi}\right|^2\int_0^\pi\int_0^\pi \sin^3\theta\cdot\sin^2(kh\sin\theta\sin\phi)\, d\theta d\phi = \frac{\eta}{2}\left|\frac{kI_0 l}{2\pi}\right|^2 I$$

$$I = \int_0^\pi \sin^3\theta\left\{\int_0^\pi \sin^2(kh\sin\theta\sin\phi)\, d\phi\right\} d\theta = \int_0^\pi \sin^3\theta\,[I_1]\, d\theta$$

where $I_1 = \int_0^\pi \sin^2(kh\sin\theta\sin\phi)\, d\phi = \frac{1}{2}\left\{\int_0^\pi d\phi - \int_0^\pi \cos(2kh\sin\theta\cdot\sin\phi\, d\phi\right\}$

$= \frac{1}{2}\left\{\pi - \int_0^\pi \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) d\phi\right\}$, where $y = 2kh\sin\theta\sin\phi$

$= \frac{\pi}{2} - \frac{1}{2}\left\{\pi - \frac{1}{2}\int_0^\pi y^2 d\phi + \frac{1}{(2\times2)!}\int_0^\pi y^4 d\phi - \frac{1}{(2\times3)!}\int_0^\pi y^6 d\phi + \cdots\right\}$

$$I_1 = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\int_0^\pi \frac{(y)^{2n}}{2n!} d\phi = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2\alpha)^{2n}}{2n!}\int_0^\pi \sin^{2n}\phi\, d\phi$$

$$I_1 = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2\alpha)^{2n}}{2n!}\left[2\int_0^{\pi/2}\sin^{2n}\phi\, d\phi\right] = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2\alpha)^{2n}}{2n!}\int_0^{\pi/2}\sin^{2n}\phi\, d\phi$$

From Mathematical Handbook of Formulas and Tables ($\leftarrow \alpha = kh\sin\theta$, $y = 2\alpha\sin\phi$)

Schaum's Outline Series, pg 96 Equation 15-30.

$$\int_0^{\pi/2}\sin^{2n}x\, dx = \frac{1\cdot3\cdot5\cdot\cdots(2n-3)(2n-1)}{2\cdot4\cdot6\cdot\cdots(2n-2)(2n)}\frac{\pi}{2}, \quad n = 1, 2, 3, 4, \ldots$$

(Continued)

4.45. (Cont'd)

Thus $I_1 = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2kh\sin\theta)^{2n}}{(2n)!} \cdot \frac{\pi}{2} \cdot A_{2n}$, where $A_{2n} = \frac{1\cdot 3\cdot 5\cdots(2n-3)(2n-1)}{2\cdot 4\cdot 6\cdots(2n-2)(2n)}$

and $I = \int_0^{\pi} \sin^3\theta \, [I_1] \, d\theta = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2kh)^{2n}}{(2n)!} \left(\frac{\pi}{2}\right) A_{2n} \int_0^{\pi} \sin^{2n+3}(\theta) d\theta$

$= \pi \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{(2kh)^{2n}}{(2n)!} \cdot A_{2n} \int_0^{\pi/2} \sin^{2n+3}\theta \, d\theta$

Using Series equation of the previous reference, or

$$\int_0^{\pi/2} \sin^{2n+3} x \, dx = \frac{2\cdot 4\cdot 6\cdot \cdots\cdot (2n-2)(2n)(2n+2)}{1\cdot 3\cdot 5\cdot \cdots\cdot (2n-1)(2n+1)(2n+3)}, \quad n = 1, 2, 3, \cdots$$

We can write that

$$I = \pi \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2kh)^{2n}}{(2n)!} (A_{2n})(A_{2n+3}), \quad A_{2n+3} = \frac{2\cdot 4\cdot 6\cdots(2n-2)(2n)(2n+2)}{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)(2n+3)}$$

However $A_{2n} \cdot A_{2n+3} = \frac{1\cdot 3\cdot 5\cdots(2n-3)(2n-1)}{2\cdot 4\cdot 6\cdots(2n-2)(2n)} \cdot \frac{2\cdot 4\cdot 6\cdots(2n-2)(2n)(2n+2)}{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)(2n+3)}$

$= (2n+2)/[(2n+1)(2n+3)]$

Therefore $I = \pi \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2kh)^{2n}}{(2n)!} \frac{(2n+2)}{(2n+1)(2n+3)}$

$$= \pi \left[ \frac{(2kh)^2}{2!}\frac{4}{3\cdot 5} - \frac{(2kh)^4}{4!}\frac{6}{5\cdot 7} + \frac{(2kh)^6}{6!}\frac{8}{7\cdot 9} - \cdots + (-1)^{n+1} \frac{(2kh)^{2n}}{(2n)!} \frac{(2n+2)}{(2n+1)(2n+3)} \right]$$

which when expanded can be written as

$$I = \pi \left\{ \frac{2}{3} - \left[ \frac{2}{3} + (2kh)^2 \left(-\frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) + (2kh)^4 \left(\frac{1}{5!} - \frac{1}{6!} + \frac{1}{7!}\right) + \cdots \pm (2kh)^{2n} \left( \frac{1}{(2n+1)!} - \frac{1}{(2n+2)!} + \frac{1}{(2n+3)!} \right) \right] \right\}$$

Recombining appropriate terms, we have that

$$I = \pi \left\{ \frac{2}{3} - \frac{1}{(2kh)} \left[ (2kh) + \sum_{n=1}^{\infty} (-1)^n \frac{(2kh)^{2n+1}}{(2n+1)!} \right] - \frac{1}{(2kh)^2} \left[ 1 - \frac{(2kh)^2}{2!} + \sum_{n=2}^{\infty} (-1)^n \frac{(2kh)^{2n}}{(2n)!} \right] \right.$$

$$\left. + \frac{1}{(2kh)^3} \left[ (2kh) - \frac{(2kh)^3}{3!} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2kh)^{2n+3}}{(2n+3)!} \right] \right\}$$

which reduces when expanded to

$$I = \pi \left[ \frac{2}{3} - \frac{\sin(2kh)}{(2kh)} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3} \right]$$

Therefore the radiated power can be written as

$$P_{rad} = \frac{\eta}{2} \left| \frac{kI_0 l}{2\pi} \right|^2 I = \eta \frac{\pi}{2} \left| \frac{I_0 l}{\lambda} \right|^2 \left[ \frac{2}{3} - \frac{\sin(2kh)}{(2kh)} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3} \right]$$

4-46. $E_\psi = C_2\sqrt{1-\sin^2\theta\sin^2\phi}\cdot[\sin(kh\cos\theta)]$, $C_2 = -\eta\frac{kI_0\ell e^{-jkr}}{2\pi r}$

a. $E_\psi(\phi=90°)\Big|_{\theta=45°} = C_2\cdot\cos\theta\cdot\sin(kh\cos\theta)\Big|_{\theta=45°} = C_2\cos(45°)\sin(\frac{kh}{\sqrt{2}}) = 0$

$\frac{kh}{\sqrt{2}} = \sin^{-1}(0) = \pm n\pi$, $n=0,1,2,3,\cdots$

Choosing the positive values and excluding the $n=0$ value, we have the smallest height of $(n=1)$

$$h = \frac{\sqrt{2}\pi}{k} = \frac{\sqrt{2}\pi}{2\pi}\lambda = \frac{\lambda}{\sqrt{2}} = 0.707\lambda$$

b. $h = \frac{\lambda}{\sqrt{2}} \Rightarrow 2kh = 2(\frac{2\pi}{\lambda})\frac{\lambda}{\sqrt{2}} = 2\sqrt{2}\pi = 8.88576$

1. $R_r = 120\pi^2(\frac{1}{50})^2\left[\frac{2}{3} - \frac{\sin(2\sqrt{2}\pi)}{2\sqrt{2}\pi} - \frac{\cos(2\sqrt{2}\pi)}{(2\sqrt{2}\pi)^2} + \frac{\sin(2\sqrt{2}\pi)}{(2\sqrt{2}\pi)^3}\right]$

$R_r = 120\pi^2(\frac{1}{50})^2[\frac{2}{3} - 0.057765 + 0.0108694 + 0.0007316] = 0.294$

2. $kh = \sqrt{2}\pi$

$$D_g = \frac{4(-0.9639)^2}{[\frac{2}{3} - 0.57765 + 0.0108694 + 0.0007316]} = \frac{4(0.9291)}{0.6205} = 5.9893$$

$D_g = 5.9893 = 7.774\,dB$

4-47. $E_\psi(\phi=90°) = C_2\cdot\cos\theta\cdot\sin(kh\cos\theta)$, $C_2 = -\eta\frac{kI_0\ell e^{-jkr}}{2\pi r}$

$E_\psi(\phi=90°)\Big|_{h=0.707\lambda} = C_2\cdot\cos\theta_n\cdot\sin(0.707\lambda k\cos\theta_n) = 0$

$\cos\theta_n = 0 \Rightarrow \theta_n = \cos^{-1}(0) = 90°$

$\sin(0.707\lambda k\cos\theta_n) = \sin(1.414\pi\cos\theta_n) = 0 \Rightarrow 1.414\pi\cos\theta_n = \sin^{-1}(0)$

$1.414\pi\cos\theta_n = \sin^{-1}(0) = \pm n\pi$, $n=0,1,2,3,\cdots$

$\theta_n = \cos^{-1}(\pm\frac{n}{1.414})$, $n=0,1,2,3,\cdots$

$n=0$: $\theta_n = \cos^{-1}(0) = 90°$

$n=\pm1$: $\theta_n = \cos^{-1}(\pm\frac{1}{1.414}) = 45°$ } for $0° \le \theta \le 90°$

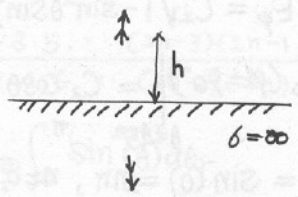
$n=\pm2$: $\theta_n = \cos^{-1}(\pm\frac{2}{1.414})$ = Does not exist.

The same holds for $n \ge 3$.

4-48.

$$H_\theta^d = j\frac{kIm\ell e^{-jkr_1}}{\eta\cdot 4\pi r}\cdot\sin\theta_1$$

$$H_\theta^r = -j\frac{kIm\ell e^{-jkr_2}}{\eta\cdot 4\pi r}\cdot\sin\theta_2 \quad ,$$

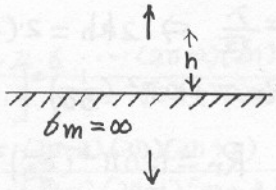



$r_1 = r - h\cos\theta$, $r_2 = r + h\cos\theta$ } phase, $r = r_1 = r_2$, for amplitude, $\theta_1 = \theta_2 = \theta$. } ⇒ Far Field

$$H_\theta = j\frac{kIm\ell e^{-jkr}}{\eta\cdot 4\pi r}\cdot\sin\theta\left[2j\sin(kh\cos\theta)\right]$$

4-49. $$E_\theta^d = j\eta\cdot\frac{kI_0\ell e^{-jkr_1}}{4\pi r_1}\cdot\sin\theta_1$$

$$E_\theta^r = -j\eta\cdot\frac{kI_0\ell e^{-jkr_2}}{4\pi r_2}\cdot\sin\theta_2$$




Far-field: $r_1 = r - h\cos\theta$, $r_2 = r + h\cos\theta$ } phase

$(r = r_1 = r_2,\ \theta = \theta_1 = \theta_2)$ amplitude

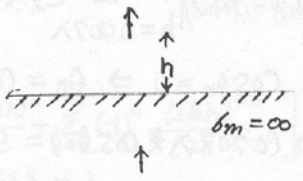
$$E_\theta = j\eta\cdot\frac{kI_0\ell e^{-jkr}}{4\pi r}\sin\theta\left[e^{jkh\cdot\cos\theta} - e^{-jkh\cos\theta}\right]$$

$$E_\theta = j\eta\cdot\frac{kI_0\ell e^{-jkr}}{4\pi r}\cdot\sin\theta\cdot\left[2j\sin(kh\cos\theta)\right]$$

4-50. $$H_\theta^{total} = H_\theta^d + H_\theta^r$$

$$H_\theta^d = \frac{jkIm\ell e^{-jkr_1}}{\eta\cdot 4\pi r_1}\cdot\sin\theta_1$$

$$H_\theta^r = \frac{jkIm\ell e^{-jkr_2}}{\eta\cdot 4\pi r_2}\cdot\sin\theta_2$$

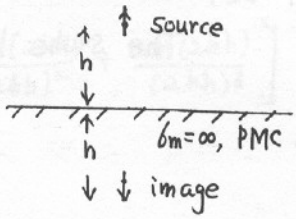



For Far-field. $r_1 \simeq r - h\cos\theta$, $r_2 \simeq r + h\cos\theta$ } phase, $(r_1 = r_2 = r)$ ⇒ Amplitude

$\theta = \theta_1 = \theta_2$

$$H_\theta^{total} = j\frac{kIm\ell e^{-jkr}}{\eta\cdot 4\pi r}\cdot\sin\theta\cdot\left[2\cos(kh\cos\theta)\right]$$

4-51. a. $$E_\theta \simeq j\eta\frac{k\ell}{4\pi}I_0\frac{e^{-jkr}}{r}\sin\theta\left[j2\sin(kh\cos\theta)\right]$$




$\sin(kh\cos 60^\circ) = 0 \rightarrow kh_n\cos 60^\circ = n\pi$, $n = 1, 2, 3, \cdots$

$$h_n = \frac{n\pi}{k\cos 60^\circ} = \frac{n\lambda}{2\cdot\cos 60^\circ} = n\lambda$$

smallest $h \rightarrow n = 1 \rightarrow h = \lambda$

(Continued)

4-51 (Cont'd)

b. $W_{av} \simeq \frac{|E_o|^2}{2\eta} \simeq \frac{\eta(kl)^2}{32\pi^2}\cdot|I_o|^2\cdot\frac{(kl)^2}{r^2}\cdot\sin^2\theta\,[4\sin^2(kh\cos\theta)]$

$U(\theta,\phi) = \lim_{r\to\infty} r^2 W_{av} = \frac{\eta}{2}\cdot\left(\frac{l}{\lambda}\right)^2\cdot|I_o|^2\cdot\sin^2\theta\cdot\sin^2(kh\cos\theta)$

$P_{rad} = \int_0^{2\pi}\int_0^{\pi/2} U(\theta,\phi)\sin\theta\,d\theta\,d\phi = \pi\eta\left(\frac{l}{\lambda}\right)^2\cdot|I_o|^2\cdot\int_0^{\pi/2}\sin^3\theta\cdot\sin^2(kh\cos\theta)\,d\theta$

$= \pi\eta\cdot\left(\frac{l}{\lambda}\right)^2\cdot|I_o|^2\cdot\left\{\frac{1}{3}+\frac{\cos(2kh)}{(2kh)^2}-\frac{\sin(2kh)}{(2kh)^3}\right\}$

$\underset{\uparrow}{=}\ \pi\eta\cdot\left(\frac{l}{\lambda}\right)^2\cdot|I_o|^2\cdot\left\{\frac{1}{3}+\frac{1}{(4\pi)^2}\right\} = \pi\eta\left(\frac{l}{\lambda}\right)^2|I_o|^2\{0.3397\}$

$kh=2\pi$

1. $D_g(\theta=45°,\phi) = \frac{4\pi U(\theta=45°,\phi)}{P_{rad}} = \frac{2\sin^2(45°)\sin^2(2\pi\cos 45°)}{0.3397}$

$= 2.74 = 4.37\text{dB}$

2. $R_r = \frac{2P_{rad}}{|I_o|^2} = 2\pi\cdot\eta\left(\frac{l}{\lambda}\right)^2\cdot\{0.3397\}$

$\frac{R_r}{\eta} = 2\pi\times10^{-4}\times 0.3397 = 2.13\times10^{-4}.$

4-52. Since $d \ll a$

$\tan\psi \simeq \frac{h_1'}{d_1} \simeq \frac{h_2'}{d_2} = \frac{h_2'}{d-d_1} \Rightarrow h_1'(d-d_1) = d_1 h_2'$

$d_1(+h_1'+h_2') = h_1'd \Rightarrow d_1 = \frac{h_1'd}{h_1'+h_2'} = \frac{5(20\times10^3)}{5+1{,}000} = 99.5$ meters

$\psi \simeq \tan^{-1}\left(\frac{h_1'}{d}\right) = \tan^{-1}\left(\frac{5}{99.5}\right) = 2.87669°$

$\frac{\sigma}{\omega\epsilon} = \frac{10^{-2}}{2\pi\times10^9(5\times10^{-9}/(36\pi))} = \frac{18}{5}\times10^{-2} = 3.6\times10^{-2} \ll 1$

Therefore the earth is a good dielectric $\Rightarrow \eta_1 \simeq \sqrt{\frac{\mu_1}{\epsilon_1}}$, $\beta_1 \simeq \omega\sqrt{\mu_1\epsilon_1}$

The divergence factor is equal to ($a = 5280$ miles $= 8.497368\times10^6$m)

$D \simeq \left[1+2\frac{h_1'h_2'}{ad\tan^3\psi}\right]^{-1/2} = \left[1+\frac{2(5)(1{,}000)}{8.497368\times10^6\times2\times10^4(0.05)^3}\right]^{-1/2} = (1+0.000463)^{-1/2}$

$\simeq 0.99977$

and the reflection coefficient equal to

$R_v = \frac{\eta_o\cos\theta_i - \eta_1\cos\theta_t}{\eta_o\cos\theta_i + \eta_1\cos\theta_t}$, where $\eta_o = \sqrt{\frac{\mu_o}{\epsilon_o}}$, $\eta_1 = \sqrt{\frac{j\omega\mu_o}{\sigma_1+j\omega\epsilon_1}} \simeq \sqrt{\frac{\mu_o}{\epsilon_1}}$

$\gamma_o\sin\theta_i = \gamma_1\sin\theta_t \Rightarrow \beta_o\sin\theta_i = \beta_1\sin\theta_t \Rightarrow \sin\theta_t = \frac{\beta_o}{\beta_1}\sin\theta_i = \sqrt{\frac{\epsilon_o}{\epsilon_1}}\sin\theta_i$

$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1-\frac{\epsilon_o}{\epsilon_1}\sin^2\theta_i} = \sqrt{1-\frac{\sin^2\theta_i}{\epsilon_r}}$

Therefore

$R_v = \frac{\cos\theta_i - \frac{\eta_1}{\eta_o}\cos\theta_t}{\cos\theta_i + \frac{\eta_1}{\eta_o}\cos\theta_t} = \frac{\cos\theta_i - \frac{1}{\sqrt{\epsilon_r}}\sqrt{1-\sin^2\theta_i/\epsilon_r}}{\cos\theta_i + \frac{1}{\sqrt{\epsilon_r}}\sqrt{1-\sin^2\theta_i/\epsilon_r}} = \frac{\epsilon_r\cos\theta_i - \sqrt{\epsilon_r-\sin^2\theta_i}}{\epsilon_r\cos\theta_i + \sqrt{\epsilon_r-\sin^2\theta_i}}$ (Continued)

4-52. (Cont'd)

$\theta_i = 90 - \psi = 90 - 2.87669° = 87.12331° \Rightarrow \sin\theta_i = 0.9987,\ \cos\theta_i = 0.0502$

Thus $R_V = \dfrac{5(0.0502) - \sqrt{5-(0.9987)^2}}{5(0.0502) + \sqrt{5-(0.9987)^2}} = \dfrac{-1.749649}{2.251649} = -0.777$

$$E_\theta \simeq j\eta \frac{I_0 e^{-jkr}}{2\pi r} \cdot \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} \cdot \left[ e^{jkh'\cos\theta} + DR_\gamma e^{-jkh'\cos\theta} \right]_{\theta=\theta_i \simeq 87.12331°}$$

$r \simeq \sqrt{d^2 + (h_2' - h_1')^2} = \sqrt{(20{,}000)^2 + (1{,}000 - 99.5)^2} = 20{,}020.26\,m = 66{,}734.207\lambda$

$h_1' = 5\,m = 16.667\lambda$, $h_2' = 1{,}000\,m = 3{,}333.3333\lambda$

$$E_\theta = j\cdot 120\pi \cdot \frac{I_0 e^{-j\frac{2\pi}{\lambda}(66{,}734.207\lambda)}}{2\pi(20{,}020.26)} \cdot \frac{\cos(\frac{\pi}{2}\cos(87.12°))}{\sin(87.12°)} \left[ e^{jkh'\cos\theta_i} + DR_V e^{-jkh'\cos\theta_i} \right]$$

$e^{-j\frac{2\pi}{\lambda}(66{,}734.207\lambda)} = e^{-j2\pi(0.207)} = e^{-j1.3} = \cos(74.52°) - j\sin(74.52°)$

$= 0.2669 - j0.9637 = 1\angle -74.52°$

$\cos[\frac{\pi}{2}\cos(87.12°)] = 0.996887$, $\sin(87.12°) = 0.99874$

$e^{jkh'\cos\theta_i} = e^{j\frac{2\pi}{\lambda}(16.667\lambda)(0.0502)} = e^{j2\pi(16.667)(0.0502)} = e^{j5.257}$

$= \cos(301.2°) + j\sin(301.2°) = 1\angle 301.2° = 0.5181 - j0.8553$

$e^{-jkh'\cos\theta_i} = 1\angle -301.2° = 0.5181 + j0.8553$

$DR_\gamma e^{-jkh'\cos\theta_i} = 0.99977(-0.777)[0.5181 + j0.8553] = -(0.4025 + j0.6644)$

Thus

$e^{jkh'\cos\theta_i} + R_\gamma D e^{-jkh'\cos\theta_i} = (0.5181 - j0.8553) - (0.4025 + j0.6644)$

$= 0.1156 - j1.5197 = 1.5241\angle -85.65°$

Therefore

$$E_\theta \simeq (1\angle 90°)(120\pi) \frac{I_0(1\angle -74.52°)(0.996887)}{2\pi(0.99874)(20{,}020.26)} (1.5241\angle -85.65°)$$

$$E_\theta \simeq 4.5592 \times 10^{-3} I_0 \angle -70.17$$

or

$$|E_\theta| = 4.5592 \times 10^{-3} |I_0| \quad \text{Volts/m}$$

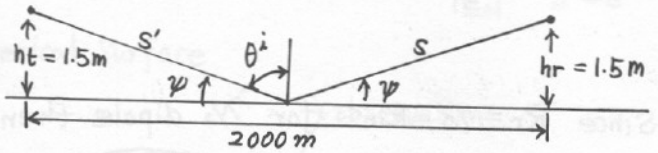
4-53. From Calibration;

$$\frac{P_r}{P_t} = \frac{C_1}{R^2} \rightarrow C_1 = \frac{P_r}{P_t} R^2 = \frac{10 \times 10^{-6}}{5} \times (10 \times 10^3)^2 = 200\ m^2$$

on asteroid

$$\frac{P_r}{P_t} = \frac{C_1}{R^2} \left| 1 + D R_v e^{-j2kh_t \cos\theta} \right|^2$$

Approximate geometry;




$$\psi = \tan^{-1}\left(\frac{1.5}{1000}\right) \simeq 1.5 \times 10^{-3} = 0.086^\circ$$

$$\theta^i = \frac{\pi}{2} - \psi \; ; \; \cos\theta^i = \sin\psi \simeq 1.5 \times 10^{-3}$$

$$\sin\theta^t = \frac{\sin\theta^i}{3} \simeq \frac{1}{3} \; ; \; \cos\theta^t \simeq \frac{2\sqrt{2}}{3}$$

$$R_v = \frac{\frac{\eta_0}{\eta_1}\cos\theta^i - \cos\theta^t}{\frac{\eta_0}{\eta_1}\cos\theta^i + \cos\theta^t} = \frac{3(1.5\times10^{-3}) - \frac{2\sqrt{2}}{3}}{3(1.5\times10^{-3}) + \frac{2\sqrt{2}}{3}} \simeq -0.9905$$

$$s' \simeq s \simeq 1000\ m \; ; \; a = 10^6\ m$$

$$D \simeq \left[1 + 2\frac{ss'}{ad\tan\psi}\right]^{-1/2} \simeq \left[1 + 2\frac{(1000)(1000)}{10^6(2000)\,1.5\times10^{-3}}\right]^{-1/2} = 0.7746$$

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{300\times10^6} = 1\ m$$

$$\left|1 + D R_v e^{-j2kh_t\cos\theta}\right|^2 \simeq \left|1 - (0.7746)(0.9905)\, e^{-j4\pi\frac{h_t}{\lambda}}\right|^2$$

$$= \left|1 - (0.7746)(0.9905)\, e^{-j4\pi}\right|^2$$

$$= 0.0541772$$

$$P_r = \frac{200}{(2\times10^3)^2}(0.0541772)(5) = 1.3544 \times 10^{-5}\ W = 13.5\ \mu W$$

4-54. $P_{rad} = 10$ Watt, $r = 3.7\times10^7$ m, $D_o = 50$dB $\Rightarrow 10^5$

a. $$D_o = \frac{4\pi\, U_{max}}{P_{rad}} = \frac{4\pi \cdot r^2|E|^2}{2\eta \cdot 10} = 10^5 \text{, (Since } U_{max} = \frac{r^2 E_{max}^2}{2\eta} \; ; \; \eta = 120\pi)$$

$$\Rightarrow E^2 = \frac{10^5 \times 2 \times 120\pi \times 10}{4\pi (3.7\times10^7)^2} = 4.4\times10^{-8}$$

$$E = 2\times10^{-4}\ V/m$$

(Continued)

4-54 (Cont'd)

b. Use Friis Transmission

$$\frac{P_r}{P_t} = \left(\frac{\lambda}{4\pi R}\right)^2 \cdot G_{ot} \cdot G_{ov} = \left(\frac{\lambda}{4\pi R}\right)^2 \cdot D_{ot} \cdot D_{or}$$

(Since we assume 100% efficiency)

At 10 GHz, $\lambda = 0.03$ m

$$\frac{P_r}{10} = \left[\frac{0.03}{4\pi \cdot (3.7\times10^{7})}\right]^2 \cdot (10{,}000)(1.643)$$

$$P_r = 6.84 \times 10^{-15}$$

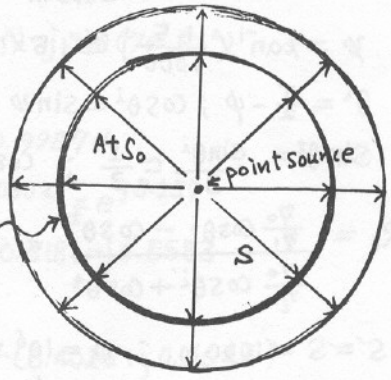
$P_{received} = \frac{V^2}{8R_{in}}$, Since $R_r = 73 = R_{in}$ for $\lambda/2$ dipole then

$$V = \sqrt{8(P_{received})(R_{in})} = 2\mu V$$

4-55. $S_o\, dA_o = S\, dA$, $\frac{S}{S_o} = \frac{dA_o}{dA}$

far zone $S = \frac{1}{2\eta}|E|^2$

$$\frac{|E|}{|E_o|} = \sqrt{\frac{dA_o}{dA}}$$




For spherical wave: $\frac{|E|}{|E_o|} = \sqrt{\frac{S_o^2}{(S+S_o)^2}} = \frac{S_o}{S+S_o}$

For plane wave: $|E|/|E_o| = 1$.

In general, it can be shown that for a wave front eikonal surface we have

$$\frac{|E|}{|E_o|} = \sqrt{\frac{\rho_1 \rho_2}{(\rho_1+S)(\rho_2+S)}}$$, $\rho_1$ and $\rho_2$ are radii of curvature of wavefront.




Wavefront at $S_o$ Wavefront at $S+S_o$

e.g. spherical wave: $\rho_1 = \rho_2 = S_o$

$$\frac{|E|}{|E_o|} = \sqrt{\frac{S_o^2}{(S+S_o)^2}} = \frac{S_o}{S+S_o}$$

plane wave: $\rho_1 = \rho_2 = \infty$

$$\frac{|E|}{|E_o|} = 1.$$

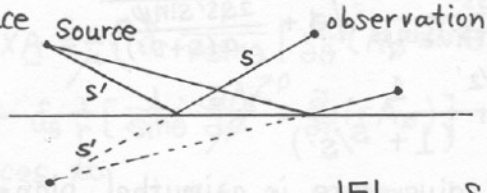
When the wave front is reflected from a surface we have

$$\frac{|E|}{|E_o|} = \sqrt{\frac{\rho_1^r \cdot \rho_2^r}{(\rho_1^r+S)(\rho_2^r+S)}} = \sqrt{\frac{1}{(1+\frac{S}{\rho_1^r})(1+\frac{S}{\rho_2^r})}}$$

(Continued)

4-55. (Cont'd) $\underline{E}$ is field at observation point.
$\underline{E}_o$ is field at reflection point.

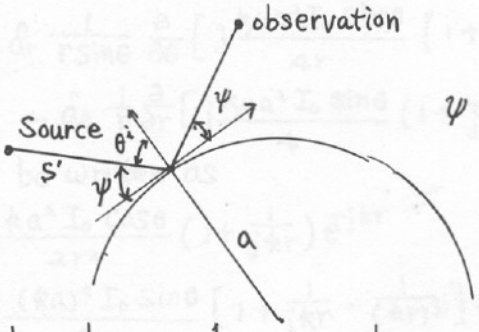
flat surface




Radius of curvature of wavefront not changed by reflection.

$\rho_1^r = \rho_2^r = \rho_1 = \rho_2 = s'$ $\quad \frac{|\underline{E}|}{|\underline{E}_o|} = \frac{s'}{s'+s} = \frac{1}{1+s/s'}$

Spherical Surface




$\psi$ : grazing angle

$\frac{1}{\rho_1^r} = \frac{1}{s'} + \frac{1}{f_1}$ ; $\frac{1}{\rho_2^r} = \frac{1}{s'} + \frac{1}{f_2}$

In physics, we always used $f_1 = f_2 = a/2$. This is not valid here because that $f$ was valid for near normal incidence ; we have near grazing incidence

$f_1 = \frac{a\cos\theta^i}{2}$ (perpendicular to the plane of incidence = elevation plane)

$f_2 = \frac{a}{2\cos\theta_i}$ : (parallel to the plane of incidence = azimuthal plane)

Thus $\frac{1}{\rho_1^r} = \frac{1}{s'} + \frac{2}{a\cos\theta_i}$ ; $\frac{1}{\rho_2^r} = \frac{1}{s'} + \frac{2\cos\theta^i}{a}$

$$\frac{|\underline{E}|}{|\underline{E}_o|} = \sqrt{\frac{1}{\left\{1+s\left(\frac{1}{s'}+\frac{2}{a\cos\theta_i}\right)\right\}\left\{1+s\left(\frac{1}{s'}+\frac{2\cos\theta^i}{a}\right)\right\}}}$$

$$= \frac{1}{\sqrt{1+\frac{s}{s'}+\frac{2s}{a\cos\theta^i}}\cdot\sqrt{1+\frac{s}{s'}+\frac{2s\cos\theta_i}{a}}}$$

$$= \frac{1}{(1+\frac{s}{s'})\sqrt{1+\frac{\frac{2ss'}{a\cos\theta_i}}{s+s'}}\sqrt{1+\frac{\frac{2ss'\cos\theta_i}{a}}{s+s'}}}$$

(Continued)

4-55 (Cont'd)

$$\frac{|E|}{|E_0|} = \frac{1}{(1+\frac{s}{s'})} \frac{1}{\sqrt{1+\frac{2ss'}{a(s+s')\sin\psi}}} \frac{1}{\sqrt{1+\frac{2ss'\sin\psi}{a(s+s')}}} \quad \leftarrow \cos\theta_i = \cos\left(\frac{\pi}{2}-\psi\right) = \sin\psi$$

$$\simeq \left[1+\frac{2ss'}{a(s+s')\sin\psi}\right]^{-1/2} \cdot \frac{1}{(1+s/s')}$$

↑ near grazing neglect divergence in azimuthal plane.

5-1. From (5-17) $\Rightarrow \underline{A} = \hat{a}_\phi A_\phi(r,\theta) = \hat{a}_\phi \, j\frac{k\mu a^2 I_0 \sin\theta}{4r}\left[1+\frac{1}{jkr}\right]e^{-jkr}$

(a) Using (3-2a) and (VII-26)

$$\underline{H} = \frac{1}{\mu}\nabla\times\underline{A} = \frac{1}{\mu}\left\{\hat{a}_r \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \cancel{\frac{\partial A_\theta}{\partial\phi}}^{\,0}\right]\right.$$
$$\left. + \hat{a}_\theta \frac{1}{r}\left[\cancel{\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}}^{\,0} - \frac{\partial}{\partial r}(rA_\phi)\right] + \hat{a}_\phi \frac{1}{r}\left[\cancel{\frac{\partial}{\partial r}(rA_\theta)}^{\,0} - \cancel{\frac{\partial A_r}{\partial\theta}}^{\,0}\right]\right\}$$

which reduces to

$$\underline{H} = \frac{1}{\mu}\left\{\hat{a}_r \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \hat{a}_\theta \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right\}$$

Using the $A_\phi$ from above

$$\underline{H} = \frac{1}{\mu}\left\{\hat{a}_r \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[j\frac{k\mu a^2 I_0 \sin\theta}{4r}\left[1+\frac{1}{jkr}\right]e^{-jkr}\right]\right.$$
$$\left. - \hat{a}_\theta \frac{1}{r}\frac{\partial}{\partial r}\left[j\frac{k\mu a^2 I_0 \sin\theta}{4}\left(1+\frac{1}{jkr}\right)e^{-jkr}\right]\right\}$$

which can be written as

$$H_r = j\frac{ka^2 I_0 \cos\theta}{2r^2}\left(1+\frac{1}{jkr}\right)e^{-jkr}$$

$$H_\theta = -\frac{(ka)^2 I_0 \sin\theta}{4r}\left[1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right]e^{-jkr}$$

$$H_\phi = 0$$

(b) Using Equation (3-10) with $\underline{J}=0$ along with the $\underline{H}$-field components from above

$$\underline{E} = \frac{1}{j\omega\epsilon}\nabla\times\underline{H} = \frac{1}{j\omega\epsilon}\left\{\hat{a}_r(0) + \hat{a}_\theta(0) + \hat{a}_\phi\frac{1}{r}\left[\frac{\partial}{\partial r}(rH_\theta) - \frac{\partial H_r}{\partial\theta}\right]\right\}$$

which reduces to

$$E_r = 0$$

$$E_\theta = 0$$

$$E_\phi = \eta\frac{(ka)^2 I_0 \sin\theta}{4r}\left[1+\frac{1}{jkr}\right]e^{-jkr}$$

The same expressions can be obtained using (3-15) with the $A_\phi$ from part a.

5-2. According to the duality theorem and the dual quantities as outlined in Table 3.2

| Electric Dipole | | Magnetic Dipole |
|---|---|---|
| $\underline{E}$ | $\Leftrightarrow$ | $\underline{H}$ |
| $\underline{H}$ | $\Leftrightarrow$ | $-\underline{E}$ |
| $\underline{I}_e$ | $\Leftrightarrow$ | $\underline{I}_m$ |
| $\epsilon$ | $\Leftrightarrow$ | $\mu$ |
| $\mu$ | $\Leftrightarrow$ | $\epsilon$ |
| $k$ | $\Leftrightarrow$ | $k$ |
| $\eta$ | $\Leftrightarrow$ | $1/\eta$ |
| $1/\eta$ | $\Leftrightarrow$ | $\eta$ |

Thus applying the above to the fields of an electric dipole, as given by (4-8a) - (4-10c), we obtain the fields of a magnetic dipole given by

$$E_r = 0$$

$$E_\theta = 0$$

$$E_\phi = -j \frac{k I_m \ell \sin\theta}{4\pi r} \left(1 + \frac{1}{jkr}\right) e^{-jkr}$$

$$H_r = \frac{1}{\eta} \frac{I_m \ell \cos\theta}{2\pi r^2} \left(1 + \frac{1}{jkr}\right) e^{-jkr}$$

$$H_\theta = j \frac{1}{\eta} \frac{k I_m \ell \sin\theta}{4\pi r} \left[1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right] e^{-jkr}$$

$$H_\phi = 0$$

which are identical to (5-20a) ~ (5-20d)

5-3. $R_r\,(1\text{ turn}) = 20\pi^2\left(\frac{c}{\lambda}\right)^4 = 20\pi^2\left(\frac{1}{5}\right)^4 = 0.31583$ ohms

$R_r\,(4\text{ turn}) = N^2 R_r(1\text{ turn}) = 4^2\cdot(0.31583) = 5.0532$ ohms

$R_L(1\text{ turn}) = R_{hf}(1\text{ turn}) = \frac{a}{b}\sqrt{\frac{\omega\mu_0}{2\sigma}} = \frac{1}{10\pi\times10^{-3}}\sqrt{\frac{2\pi\times10^7\cdot(4\pi\cdot10^{-7})}{2\cdot(5.7\times10^7)}}$

$R_L = R_{hf} = 0.0265$

$R_L\,(4\text{ turn}) = R_{ohmic} = \frac{Na}{b}R_s\left(\frac{R_p}{R_0}+1\right)$

$R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}} = \sqrt{\frac{2\pi\times10^7\times(4\pi\times10^{-7})}{2(5.7\times10^7)}} = 8.3223\times10^{-4}$

$R_0 = \frac{NR_s}{2\pi b} = \frac{4\cdot(8.3223\times10^{-4})}{2\pi(10^{-3})} = 0.5298$

$\frac{R_p}{R_0} \simeq 0.5$ from Fig 5.26

Thus $R_L = R_{ohmic} = \frac{4(8.3223\times10^{-4})}{4\pi\times10^{-3}}(0.5+1) = 0.15724$

and $e_{cd}\,(1\text{ turn}) = 100\cdot R_r/(R_r+R_L) = \frac{0.3158\times100}{0.3158+0.0265} = 92.26 = 92.26\%$

$e_{cd}\,(4\text{ turn}) = 100\cdot R_r/(R_r+R_L) = \frac{5.0532(100)}{5.0532+0.15724} = 96.98\%$

5-4. $H_\theta = -\frac{\pi S I_0 e^{-jkr}}{\lambda^2 r}\sin\theta$ where $S = \pi a^2$

$E_\phi = -\eta H_\theta = \eta\frac{\pi S I_0 e^{-jkr}}{\lambda^2 r}\sin\theta$

$\underline{W}_{ave} = \frac{1}{2}\mathrm{Re}(\underline{E}\times\underline{H}^*) = \frac{1}{2}\mathrm{Re}(\hat{a}_\phi E_\phi \times \hat{a}_\theta H_\theta^*) = \frac{1}{2}\mathrm{Re}(-\hat{a}_\phi \eta H_\theta \times \hat{a}_\theta H_\theta^*)$

$\underline{W}_{ave} = \hat{a}_r\frac{1}{2}\mathrm{Re}(\eta|H_\theta|^2) = \hat{a}_r\frac{\eta}{2}|H_\theta|^2 = \hat{a}_r\frac{\eta}{2}\left|\frac{\pi S I_0}{\lambda^2}\right|\frac{\sin^2\theta}{r^2} = \hat{a}_r W_r$

$P_{rad} = \oint\!\!\oint_{S_0}\underline{W}_{ave}\cdot d\underline{s} = \int_0^{2\pi}\int_0^{\pi}\hat{a}_r W_r\cdot\hat{a}_r r^2\sin\theta\,d\theta\,d\phi = 2\pi\int_0^{\pi} W_r r^2\sin\theta\,d\theta$

$= \pi\eta\left|\frac{\pi S I_0}{\lambda^2}\right|^2\int_0^{\pi}\sin^3\theta\,d\theta = \frac{4\pi\eta}{3}\left|\frac{(\pi a)^2 I_0}{\lambda^2}\right|^2$

$= \eta\frac{\pi}{12}(ka)^4|I_0|^2$

5-5. $\underline{A} = \hat{a}_\phi \, j\frac{k\mu a^2 I_o \sin\theta}{4r}\left(1+\frac{1}{jkr}\right)e^{-jkr} \simeq \hat{a}_\phi \, j\frac{k\mu a^2 I_o e^{-jkr}}{4r}\sin\theta$

from equation (5-17) and $r \to$ large.

Using (3-58a)

$E_r \simeq E_\theta \simeq 0$

$E_\phi \simeq -j\omega A_\phi = -j\omega\left(j\frac{k\mu a^2 I_o e^{-jkr}}{4r}\sin\theta\right) = \eta\frac{\pi S I_o e^{-jkr}}{\lambda^2 r}\sin\theta$

where $S = \pi a^2$, $\eta = \sqrt{\mu/\epsilon}$

also using (3-58b)

$H_r \simeq H_\phi \simeq 0$

$H_\theta \simeq j\frac{\omega}{\eta}A_\phi = j\frac{\omega}{\eta}\left(j\frac{\mu k a^2 I_o e^{-jkr}}{4r}\sin\theta\right) = -\frac{\pi S I_o e^{-jkr}}{\lambda^2 r}\sin\theta$

5-6. a. $R_r = 20\cdot\pi^2\left(\frac{C}{\lambda}\right)^4 = 20\cdot\pi^2\left(\frac{2\pi a}{\lambda}\right)^4$, $\lambda = \frac{3\times10^8}{10^7} = 30$ m

$0.73 = 20\pi^2\left(\frac{2\pi a}{\lambda}\right)^4 \Rightarrow a = 0.03924\lambda = 1.177$ meters

b. $0.73N^2 = 300 \Rightarrow N = 20.272 \simeq 20$

$R_r$ (20 turns) $= 0.73(20)^2 = 292$

c. $P_L = A_{em}\cdot W_i\cdot e_z = \frac{\lambda^2}{4\pi}D_o\epsilon_x(10^{-6}) = \frac{\lambda^2}{4\pi}\left(\frac{3}{2}\right)(1-|\Gamma|^2)\cdot10^{-6}$

$= \frac{(30)^2}{4\pi}\left(\frac{3}{2}\right)\cdot\left(1-\left|\frac{292-300}{292+300}\right|^2\right)\cdot10^{-6} = 0.1074\times10^{-3}$ watts

5-7. $a = \lambda/30$, $b = \lambda/300$, $2C = \lambda/100 \Rightarrow C = \lambda/200$, $N = 6$, $f = 5\times10^7$ Hz

a. Since $a = \lambda/30 \ll \lambda$

$D_o = 1.5 = 1.761$ dB

b. $R_r = 20\pi^2\left(\frac{C}{\lambda}\right)^4 = 20\pi^2\left(\frac{\pi}{15}\right)^4 = 20\pi^2(1.924\times10^{-3}) = 0.3798$ ohms

$C = 2\pi a = 2\pi\left(\frac{\lambda}{30}\right) = \frac{\pi}{15}\lambda$

$R_r$ (Single turn) $= 0.3798$ ohms

$R_r$ (6 turns) $= 13.673$ ohms

$R_L = \frac{Na}{b}\cdot R_s\cdot\left(\frac{R_p}{R_o}+1\right)$

$C/b = \frac{\lambda/200}{\lambda/300} = \frac{3}{2} = 1.5 \Rightarrow \frac{R_p}{R_o} = 0.65$

(Continued)

5-7. (Cont'd) $R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}} = \sqrt{\frac{2\pi f(4\pi\times10^{-7})}{2(5.7\times10^7)}} = \sqrt{\frac{4\pi^2 f}{5.7}}\times10^{-7}$

$= \sqrt{\frac{4\pi^2(50\times10^6)}{5.7}}\times10^{-7} = 2\pi\sqrt{\frac{50}{5.7}}\times10^{-4} = 18.609\times10^{-4}$

$R_L = 6\left(\frac{\lambda/30}{\lambda/300}\right)\cdot 18.609\times10^{-4}(0.65+1)$

$= 6(10)(18.609)(1.65)\times10^{-4} = 1,842.31\times10^{-4}$

$R_L$ (6 turns) $= 0.184231$

(Single) $R_L = \frac{\lambda/30}{\lambda/300}\sqrt{\frac{2\pi f(4\pi\times10^{-7})}{2\cdot(5.7\times10^7)}} = 10\sqrt{\frac{4\pi^2 f}{5.7}}\times10^{-7}$

$= 2\pi(10)\sqrt{\frac{50}{5.7}}\times10^{-4} = 186.0919\times10^{-4}$

(6 turns) $R_L = 186.0919\cdot(6)(1.65)\times10^{-4} = 1,842.31\times10^{-4}$

$e_{cd} = \frac{R_r}{R_r+R_L} = \frac{13.673}{13.673+0.184231}\times100 = 98.67\%$

c. $|\Gamma| = \left|\frac{(R_r+R_L)-50}{(R_r+R_L)+50}\right| = \left|\frac{13.857-50}{13.857+50}\right| = \left|\frac{-36.14277}{63.857}\right| = 0.566$

$e_r = (1-|\Gamma|^2)\times100 = (1-|0.566|^2)\times100 = (1-0.32)\times100 = 68\%$

d. $G_o = e_{cd}D_o = (0.9867)D_o = (0.9867)(1.5)$

$G_o = 1.48005$. (← Total Maximum gain does not include the reflection loss)

5-8. $f = 30$ MHz $\rightarrow \lambda = 10$ m, $ka = \frac{2\pi}{10}(0.15) = 0.03\pi = 0.09425$ (rad)

$R_r = N^2\frac{\pi\eta_0}{6}(ka)^4 = 64\times\frac{\pi^2 120}{6}\times(0.03\pi)^4 = 0.9968\ \Omega$

$\delta = \frac{1}{\sqrt{\pi f\mu_0\sigma}} = \frac{1}{\sqrt{\pi\times30\times10^6\times4\pi\times10^{-7}\times5.7\times10^7}} = 1.217\times10^{-5}$ m $\ll b$

1-turn: $R_L = \frac{a}{\sigma\cdot b\delta} = \frac{0.15}{5.7\times10^7\times0.001\times1.217\times10^{-5}} = 0.2162\ \Omega$

8-turn: $R_L = 8\times R_L(1\text{turn})\times\left(\frac{R_p}{R_o}+1\right)$, $c/b = 1.8 \Rightarrow \frac{R_p}{R_o} = 0.5$

$\therefore R_L = 8\times(0.2162)\times1.5 = 2.594\ \Omega$

$e_{cd} = \frac{R_r}{R_r+R_L} = \frac{0.9968}{0.9968+2.594} = 0.278 = 27.8\%$

5-9. A small loop is equivalent to an infinitesimal magnetic dipole whose axis is vertical to the plane of the loop provided that $I_m \ell = j S \omega \mu I_0$ where $I_m$ is the current of the magnetic dipole, $\ell$ is its length, $S = \pi a^2$ = area of the loop, and $I_0$ = current of the loop. Thus the fields of the loop can be obtained from those of an equivalent magnetic dipole.

a. When the loop lies on the $yz$-plane, the fields can be written, using (5-20a)-(5-20d) for $r \to$ large, as

$$E_\chi \simeq -j \frac{k I_m \ell e^{-jkr}}{4\pi r} \sin\psi, \quad H_\psi \cong \frac{E_\chi}{\eta}$$

where $$\sin\psi = \sqrt{1-\cos^2\psi} = \sqrt{1-|\hat{a}_x \cdot \hat{a}_r|^2} = \sqrt{1-\sin^2\theta\cos^2\phi}$$

$$I_m \ell = j S \omega \mu I_0 = j \pi a^2 \omega \mu I_0$$

Thus $$E_\chi = \frac{a^2 \omega \mu k I_0 e^{-jkr}}{4r} \sqrt{1-\sin^2\theta\cos^2\phi}, \quad H_\psi = \frac{E_\chi}{\eta}$$

b. Since the field pattern of the antenna is $\sin\psi$ where $\psi$ is measured from its axis, its directivity is the same of a small electric dipole or small loop. Thus $D_0 = \frac{3}{2}$.

5-10. Using the results of Problem 5-9

a. $$E_\chi \cong \frac{a^2 \omega \mu k I_0 e^{-jkr}}{4r} \sqrt{1-|\hat{a}_y \cdot \hat{a}_r|^2} = \frac{a^2 \omega \mu k I_0 e^{-jkr}}{4r} \sqrt{1-\sin^2\theta\sin^2\phi}$$

$$H_\psi \cong \frac{E_\chi}{\eta}$$

b. Directivity $= D_0 = \frac{3}{2}$

5-11. Using the computer program at the end of Chapter 5.

a. $a = \lambda/50 = 0.02\lambda$

$D_o = 1.4988 = 1.7575\,dB$, $R_r = 0.049$ ohms

b. $a = \lambda/10 = 0.1\lambda$

$D_o = 1.4699 = 1.6731\,dB$, $R_r = 28.41$ ohms

c. $a = \lambda/4 = 0.25\lambda$

$D_o = 1.2969 = 1.1291\,dB$, $R_r = 723.938$ ohms

d. $a = \lambda/2 = 0.5\lambda$

$D_o = 1.7968 = 2.5449\,dB$, $R_r = 2,202.528$ ohms

5-12. According to (5-54b)

$$E_\phi \simeq \frac{ak\eta I_o e^{jkr}}{2r} J_1(ka\sin\theta) \sim J_1(ka\sin\theta)$$

Therefore the nulls of the pattern occur when

$$J_1(ka\sin\theta_n) = 0 \Rightarrow ka\sin\theta_n = 0,\ 3.84,\ 7.01,\ 10.19,\ \cdots$$

Excluding $\theta = 0$

$$\theta_n = \begin{cases} \sin^{-1}\left(\frac{3.84}{ka}\right) = \sin^{-1}\left[\frac{3.84}{2\pi(1.25)}\right] = \sin^{-1}(0.4889) = 29.27^\circ \\ \sin^{-1}\left(\frac{7.01}{ka}\right) = \sin^{-1}\left[\frac{7.01}{2\pi(1.25)}\right] = \sin^{-1}(0.8925) = 63.19^\circ \end{cases}$$

5-13. Since $E_\phi \sim J_1(ka\sin\theta)$

a. $E_\phi|_{\theta=0} = J_1(ka\sin\theta)|_{\theta=0} = J_1(0) = 0$

$E_\phi|_{\theta=\pi/2} = J_1(ka\sin\theta)|_{\theta=90^\circ} = J_1(ka) = 0 \Rightarrow ka = 3.84$

Thus $a = \frac{3.84}{k} = \frac{3.84\lambda}{2\pi} = 0.61115\lambda$

b. Since $a = 0.61115\lambda > 0.5\lambda$, use large loop approximation. According to (5-63a) $R_r = 60\pi^2(C/\lambda) = 60\pi^2\left(\frac{2\pi a}{\lambda}\right) = 60\pi^2(2\pi(0.61115))$

$= 2,273.94$

c. The directivity is given by (5-63b), or

$$D_o = 0.682\left(\frac{C}{\lambda}\right) = 0.682\left(\frac{2\pi a}{\lambda}\right) = 0.682(2\pi)(0.61115) = 2.619$$

5-14. $E_\phi \sim J_1(ka\sin\theta)$

a. $E_\phi|_{\theta=30^\circ} = J_1(ka\sin\theta)|_{\theta=30^\circ} = J_1\left(\frac{ka}{2}\right) = 0 \Rightarrow \frac{ka}{2} = 3.84$

From the Table for $J_1(x)$ in Appendix V . Thus

$$a = \frac{2(3.84)}{k} = \frac{2(3.84)}{2\pi}\lambda = 1.222\lambda$$

b. $E_\phi|_{max} = E_\phi|_{ka\sin\theta=1.84} = J_1(1.84) = 0.58152 = -4.709\,dB$

$$E_\phi|_{\theta=90^\circ} = J_1(ka) = J_1\left[\frac{2\pi}{\lambda}(1.222\lambda)\right] = J_1(7.678) = 0.175$$
$$= -15.139\,dB$$

Thus $\Delta E = E_\phi|_{\theta=90^\circ} - E_\phi|_{max} = -15.139 - (-4.709) = -10.43\,dB$

5-15. $E_\phi \sim J_1(ka\sin\theta)$

a. According to the Table for $J_1(x)$ in Appendix V

$J_1(x) = 0$ when $x = 0, 3.84, 7.01, 10.19, \cdots$

Since we want a null in the plane of the loop ($\theta = 0^\circ$) and two additional ones for $0^\circ \le \theta \le 90^\circ$, then

$ka\sin\theta|_{max} = ka\sin\theta|_{\theta=90^\circ} = ka = 7.01$

Thus $a = \frac{7.01}{k} = \frac{7.01}{2\pi}\cdot\lambda = 1.1157\lambda$

b. The nulls will occur at

$\theta = 0^\circ$ and $180^\circ$

$\theta = 90^\circ$

and $ka\sin\theta|_{a=1.1157\lambda} = 3.84$

$$\Rightarrow \theta = \sin^{-1}\left[\frac{3.84}{2\pi(1.1157)}\right] = 33.21^\circ$$

and $\theta = 180^\circ - 33.21^\circ = 146.79^\circ$

5-16. $\underline{E} = \hat{a}_\phi C_1 J_1(ka\sin\theta)$ where $C_1$ is a constant $\Rightarrow \hat{\rho}_w = \hat{a}_\phi$

and $PLF = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = |\hat{a}_\phi \cdot \hat{\rho}_a|^2$

By inspection, the PLF is maximized if the probe antenna is also linearly polarized in the $\phi$ direction. This can be accomplished by using as a probe antenna another loop antenna so that

$\hat{\rho}_a = \hat{a}_\phi$ and $PLF = |\hat{a}_\phi \cdot \hat{a}_\phi|^2 = 1$.

It can also be accomplished by using a linear dipole as a probe antenna with its length parallel to the plane of the loop and tangent to its curvature. Some specific examples would be [using the transformation of VII-7b ]

$$\hat{\rho}_a = \hat{a}_x|_{\phi=\phi_0^\circ} \Rightarrow PLF = |\hat{a}_\phi \cdot \hat{a}_x|_{\phi=\phi_0^\circ} = |\hat{a}_\phi \cdot (\hat{a}_\rho \cos\phi - \hat{a}_\phi \sin\phi)|^2_{\phi=\phi_0^\circ}$$

$$= |\hat{a}_\phi \cdot (-\hat{a}_\phi)|^2 = 1$$

$$\hat{\rho}_a = \hat{a}_y|_{\phi=0^\circ} \Rightarrow PLF = |\hat{a}_\phi \cdot \hat{a}_y|_{\phi=0^\circ} = |\hat{a}_\phi \cdot (\hat{a}_\rho \sin\phi + \hat{a}_\phi \cos\phi)|^2_{\phi=0^\circ}$$

$$= |\hat{a}_\phi \cdot \hat{a}_\phi|^2 = 1.$$

and many others.

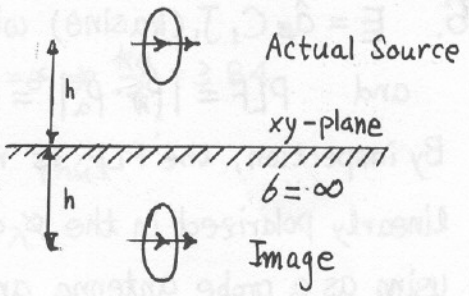
5-17. A very small loop of constant current is equivalent to a magnetic dipole. Since the loop is placed for both parts (a and b) perpendicular to the xy-plane (the plane of the loop is perpendicular to the xy-plane), the axis of the linear magnetic dipole will also be parallel to the xy-plane. Therefore according to Figure 4.12a, the image of the horizontal magnetic dipole will be as shown in this figure. In turn the array factor for both parts (a and b) of this problem will be the same as that of the vertical electric dipole of Figure 4.13 or

$$AF = 2\cdot\cos(kh\cos\theta)$$

(Continued)

5-17. (Cont'd)

Since the actual source and the image are oriented in the same direction.
Therefore according to (5-27a)-(5-27c)




a. Plane of the loop is parallel to the $xz$-plane

$$E_\chi = \eta\frac{(ka)^2 I_0 e^{-jkr}}{4r}\sin\psi\,(AF), \quad \sin\psi=\sqrt{1-\cos^2\psi}=\sqrt{1-|\hat{a}_y\cdot\hat{a}_r|^2}$$

$$=\sqrt{1-\sin^2\theta\sin^2\phi}$$

$$=\eta\frac{(ka)^2 I_0 e^{-jkr}}{4r}\sin\psi\,[2\cos(kh\cos\theta)]$$

$$E_\chi = \eta\frac{(ka)^2 I_0 e^{-jkr}}{2r}\cos(kh\cos\theta)\sqrt{1-\sin^2\theta\sin^2\phi}$$

$$H_\psi = -\frac{E_\chi}{\eta}$$

b. Plane of the loop is parallel to the $yz$-plane.

The fields for this problem are the same as those in part a. above except that

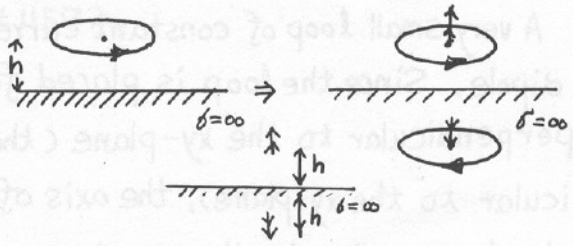
$$\sin\psi=\sqrt{1-\cos^2\psi}=\sqrt{1-|\hat{a}_x\cdot\hat{a}_r|^2}=\sqrt{1-\sin^2\theta\cos^2\phi}$$

5-18.

a. $$E_\phi = \eta\frac{(ka)^2\cdot I_0\cdot e^{-jkr}}{4r}\sin\theta$$

$$|AF| = |2j\sin(kh\cos\theta)|$$

$$E_\phi = \eta\frac{\pi S I_0 e^{-jkr}}{\lambda^2 r}\cdot\sin\theta, \quad S=\pi a^2$$




$$(E_\phi)_t = E_\phi\,(AF) = \eta\frac{\pi S I_0 e^{-jkr}}{\lambda^2 r}\sin\theta\cdot[2j\sin(kh\cos\theta)],$$ ← above ground plane total field.

b. $h=\lambda$, $kh=2\pi$

$$\sin\theta\cdot[2j\sin(2\pi\cos\theta)]=0, \quad \sin(2\pi\cos\theta)=0, \quad 2\pi\cos\theta=n\pi. \quad n=0,1,2$$

$$\theta_n=0^\circ, \quad \cos\theta_n=\frac{n}{2}, \quad n=0,1,2. \rightarrow \theta_n=90^\circ,$$

$$\theta_0=\cos^{-1}(0)=90^\circ$$

$$\theta_1=\cos^{-1}\left(\tfrac{1}{2}\right)=60^\circ$$

$$\theta_2=\cos^{-1}(1)=0^\circ.$$

(Continued)

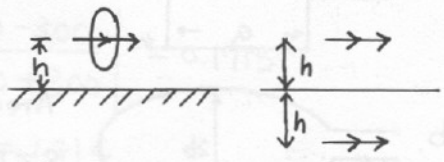
5-18. (Cont'd)

c. $(E_\phi)_t = C\sin\theta\sin(kh\cos\theta)\big|_{\theta=60^\circ} = 0 = C\cdot\left(\frac{\sqrt{3}}{2}\right)\cdot\sin\left(\frac{2\pi}{\lambda}\cdot h\cdot\frac{1}{2}\right) = C\cdot\frac{\sqrt{3}}{2}\cdot\sin\left(\frac{\pi h}{\lambda}\right)$

$\sin\left(\frac{\pi h}{\lambda}\right)=0 \Rightarrow \frac{\pi h}{\lambda} = \sin^{-1}(0) = n\pi,\ n=0,1,2,3,\cdots$

$\frac{h}{\lambda} = \pm n \Rightarrow$ Physical Nonzero height $\Rightarrow h = n\lambda,\ n=1,2,3,\cdots$

5-19. a. Array Factor $= 2\cos(kh\cos\theta)$




b. $AF = 2\cos(kh\cos\theta_n) = 0$

$\Rightarrow kh\cos\theta_n = \cos^{-1}(0) = n\pi/2,\ n=\pm1,\pm3,\pm5,\cdots$

$\theta_n = \cos^{-1}\left[\frac{n\pi/2}{kh}\right] = \cos^{-1}\left(\frac{\frac{n\pi}{2}}{\frac{2\pi}{\lambda}h}\right) = \cos^{-1}\left(\frac{n\lambda}{4h}\right)\Big|_{h=\lambda/2} = \cos^{-1}\left(\frac{n}{2}\right)$

$\theta_1 = \cos^{-1}(\pm\frac{1}{2}) = 60^\circ,\ \theta_3 = \cos^{-1}(\pm\frac{3}{2}) =$ does not exist.

5-20. From Problem 5.17 (a)

$$E_\chi\Big|_{\substack{\phi=90^\circ\\ \theta=45^\circ}} = C_1\cos(kh\cos\theta)\sqrt{1-\sin^2\theta\sin^2\phi}\,\Big|_{\substack{\theta=45^\circ\\ \phi=90^\circ}}$$

$$= C_1\cos(kh\cos\theta)\cos(\theta)\Big|_{\theta=45^\circ}$$

$$= 0.707\cdot C_1\cdot\cos\left(\frac{kh}{\sqrt{2}}\right) = 0 \Rightarrow \frac{kh}{\sqrt{2}} = \cos^{-1}(0) = \frac{\pi}{2}n,\ n=1,3,5,\cdots$$

For the smallest height

$$\frac{kh}{\sqrt{2}} = \frac{\pi}{2} \Rightarrow h = \frac{\sqrt{2}}{2}\frac{\pi}{k} = \frac{\sqrt{2}}{4}\lambda = 0.3535\lambda$$

5-21. a. $R_L = \frac{a}{b}\sqrt{\frac{\omega\mu_0}{2\sigma}} = \frac{1}{20\cdot(10^{-4})}\cdot\sqrt{\frac{2\pi(3\times10^8)(4\pi\times10^{-7})}{2\cdot5.7\times10^7}} = \frac{\pi}{20}\sqrt{\frac{12}{5.7}}\times10$

$= 2.27915$ ohms

b. $R_r = 120\cdot\pi\cdot(\frac{2}{3}\pi)\cdot\left(\frac{kS}{\lambda}\right)^2 = 120\cdot\pi(\frac{2}{3}\pi)\left(\frac{2\pi^2}{(20)^2}\right)^2 = 80\cdot\frac{4\cdot\pi^6}{(400)^2} = 1.92278$ ohms

$(\leftarrow S = \pi(\frac{1}{20})^2)$

c. inductive reactance $X_A = \omega L_A$

$L_A = \mu_0 a\cdot\left[\ln\left(\frac{8a}{b}\right)-2\right] = 4\pi\times10^{-7}\cdot(\frac{\lambda}{20})\cdot\left[\ln\left(\frac{1}{20}\cdot\frac{1}{10^{-4}}\right)-2\right] = 2.648\times10^{-7}$

$(\leftarrow a = \frac{\lambda_0}{20},\ b = 10^{-4}\lambda_0)\ \leftarrow \lambda = 1\text{m},\ f = 3\times10^8$

$X_A = 2\pi f\cdot L_A = 2\pi\cdot(3\times10^8)\cdot(2.648\times10^{-7}) = 499.158$

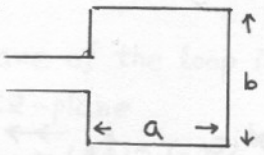
$\therefore X_A >> (R_L \text{ or } R_r)$

5-22. From equation (5-24)

$$R_r = \eta\left(\frac{\pi}{6}\right)(k^2a^2)^2 = \eta\cdot\frac{2\pi}{3}\cdot\left(\frac{kS}{\lambda}\right)^2 = 120\pi\cdot\frac{2\pi}{3}\cdot\left(\frac{2\pi S}{\lambda^2}\right)^2$$

$$= 120\cdot\frac{2}{3}\cdot 4\cdot\pi^4\left(\frac{S}{\lambda^2}\right)^2 = 31170.909\cdot\frac{S^2}{\lambda^4}$$

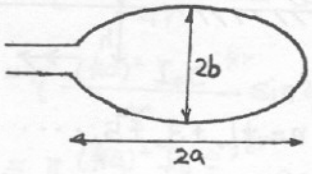
a.




AREA

$S = ab$, $R_r = 31170.909\cdot\frac{a^2b^2}{\lambda^4}$

b.




Area

$S = \pi ab$, $R_r = 31170.909\cdot\frac{\pi^2a^2b^2}{\lambda^4}$

5-23. In Far-Field ($kr \gg 1$) region

$$E_a = E_\phi\hat{a}_\phi = -j\eta\frac{kI_{in}}{4\pi r}\cdot l_e\cdot e^{-jkr} \qquad (\rightarrow l_e : \text{effective length})$$

$$E_\phi \simeq \eta\frac{k^2a^2I_0e^{-jkr}}{4r}\cdot\sin\theta = \eta\frac{\pi S I_0 e^{-jkr}}{\lambda^2 r}\sin\theta \qquad \left(\leftarrow \frac{\pi}{\lambda^2} = \frac{k^2}{4\pi}\right)$$

$$= -j\eta\frac{kI_0(jkS\cdot\sin\theta)}{4\pi r}e^{-jkr}$$

$$\therefore\ l_e = jkS\sin\theta\,\hat{a}_\phi$$

5-24. $C = 2\pi a = 1.4\lambda \Rightarrow a = \frac{1.4\lambda}{2\pi} = 0.2228\lambda$

$\Omega = 2\ln(2\pi\frac{a}{b}) = 2\ln(2\pi\frac{0.2228}{0.01555}) = 9.0$

a. From Figure 5.11

$Z_{in} = R_{in} + jX_{in} = 220 - j40$

b. $|\Gamma| = \left|\frac{Z_{in} - Z_c}{Z_{in} + Z_c}\right| = \left|\frac{220 - j40 - 300}{220 - j40 + 300}\right| = 0.1715$

$VSWR = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.1715}{1 - 0.1715} = 1.414$

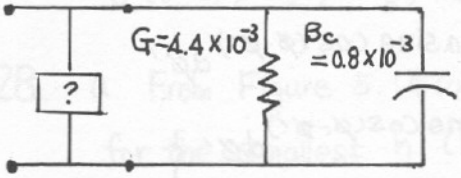
C. $Y_{in} = \frac{1}{Z_{in}} = \frac{1}{220 - j40} = (4.4 + j0.8)\times 10^{-3} = G + jB_c$

To resonate the circuit, the unknown element must have an inductive admittance of

$Y_{unknown} = -j0.0008 = -j\frac{1}{\omega L} \Rightarrow L = \frac{1}{0.8\times 10^{-3}(2\pi f)}$

$= \frac{1}{0.8\times 10^{-3}(2\pi\times 10^8)}$

$L = \frac{10^{-5}}{1.6\pi} = 1.989\times 10^{-6}\ h$




There fore the unknown element across the terminals of the loop must be an inductor of $L = 1.989\times 10^{-6}$ henries

5-25. a. From Figure 5.11 (a, b)

$Z_{in} = 90 - j110$

b. Inductor; $X_L = +110 = \omega L = 2\pi f L$

$L = \frac{110}{2\pi f} = \frac{110}{2\pi\cdot(10^9)} = \frac{110}{2\pi}\times 10^{-9}$

c. $Z_{in} = 90$

$|\Gamma| = \left|\frac{Z_{in} - Z_c}{Z_{in} + Z_c}\right| = \frac{90 - 78}{90 + 78} = \frac{12}{168} = 0.0714$

$VSWR = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.0714}{1 - 0.0714} = \frac{1.0714}{0.9285} = 1.1538$

5-26. a. From Figure 5.11(b) $Z_{in} = R_{in} + jX_{in} = R_{in} \Rightarrow X_{in} = 0$ when

$$\Omega = 2\ln(2\pi\frac{a}{b}) \simeq \begin{cases} 12 \Rightarrow 2\pi(a/b) = e^{6} = 403.429 \Rightarrow \frac{a}{b} = 64.21 \\ 11 \Rightarrow 2\pi(a/b) = e^{5.5} = 244.692 \Rightarrow \frac{a}{b} = 38.94 \\ 10 \Rightarrow 2\pi(a/b) = e^{5} = 148.413 \Rightarrow \frac{a}{b} = 23.62 \\ 9 \Rightarrow 2\pi(a/b) = e^{4.5} = 90.017 \Rightarrow \frac{a}{b} = 14.33 \end{cases}$$

b. These occur when the smallest circumference of the loop is <from Figure 5.11(b)>

$\Omega = 12 \Rightarrow C = 2\pi a \simeq 1.08\lambda \Rightarrow a = 0.1719\lambda \Rightarrow b = 0.1719\lambda/64.21 = 2.68\times10^{-3}\lambda$

$\Omega = 11 \Rightarrow C = 2\pi a \simeq 1.10\lambda \Rightarrow a = 0.175\lambda \Rightarrow b = 0.175\lambda/38.94 = 4.496\times10^{-3}\lambda$

$\Omega = 10 \Rightarrow C = 2\pi a \simeq 1.14\lambda \Rightarrow a = 0.1814\lambda \Rightarrow b = 0.1814\lambda/23.62 = 7.68\times10^{-3}\lambda$

$\Omega = 9 \Rightarrow C = 2\pi a \simeq 1.28\lambda \Rightarrow a = 0.2037\lambda \Rightarrow b = 0.2037\lambda/14.33 = 14.216\times10^{-3}\lambda$

5-27. $I(\phi) = I_0\cos\phi$

a. $$\bar{A}(\bar{F}) = \frac{\mu I_0}{4\pi} a\int_0^{2\pi} \hat{a}_\phi \cos\phi' \frac{e^{-jkR}}{R} d\phi' \simeq \frac{\mu I_0}{4\pi} a \frac{e^{-jkr}}{r}\int_0^{2\pi} \hat{a}_\phi \cos\phi' e^{jk\hat{a}_r\cdot\bar{F}'} d\phi'$$

$$= \frac{\mu I_0}{4\pi} a \frac{e^{-jkr}}{r}\left\{-\hat{a}_x\int_0^{2\pi}\cos\phi'\sin\phi' e^{jka\sin\theta\cos(\phi-\phi')} d\phi' + \hat{a}_y\int_0^{2\pi}\cos^2\phi' e^{jka\sin\theta\cos(\phi-\phi')} d\phi'\right\}$$

$$= \frac{\mu I_0 a}{8\pi} \frac{e^{-jkr}}{r}\left\{-\hat{a}_x\int_0^{2\pi}\sin(2\phi') e^{jka\sin\theta\cos(\phi-\phi')} d\phi' + \hat{a}_y\int_0^{2\pi}(\cos(2\phi')+1) e^{jka\sin\theta\cos(\phi-\phi')} d\phi'\right\}$$

$$= \frac{\mu I_0 a}{4} \frac{e^{-jkr}}{r}\left\{\hat{a}_x J_2(ka\sin\theta)\sin 2\phi - \hat{a}_y J_2(ka\sin\theta)\cos 2\phi + \hat{a}_y J_0(ka\sin\theta)\right\}$$

$$= \frac{\mu I_0 a}{2} \frac{e^{-jkr}}{r}\left\{-\hat{a}_\phi J_2(ka\sin\theta)\cos\phi + \hat{a}_y \frac{1}{2}[J_2(ka\sin\theta) + J_0(ka\sin\theta)]\right\}$$

$$= \frac{\mu I_0 a}{2} \frac{e^{-jkr}}{r}\left\{-\hat{a}_\phi J_2(ka\sin\theta)\cos\phi + \hat{a}_y \frac{J_1(ka\sin\theta)}{ka\sin\theta}\right\}$$

(Continued)

5-27 (cont'd)

$$A_\phi \simeq \frac{\mu I_0 a}{2}\frac{e^{-jkr}}{r}\left\{J_2(ka\sin\theta) - \frac{J_1(ka\sin\theta)}{ka\sin\theta}\right\}\cos\phi$$

$$= \frac{\mu I_0 a}{2}\frac{e^{-jkr}}{r} J_1'(ka\sin\theta)\cos\phi$$

$$A_\theta \simeq \frac{\mu I_0 a}{2}\frac{e^{-jkr}}{r}\frac{J_1(ka\sin\theta)}{ka\sin\theta}\cos\theta\sin\phi$$

$$E_\phi \simeq \frac{j\eta ka}{2} I_0 \frac{e^{-jkr}}{r} J_1'(ka\sin\theta)\cos\phi$$

$$E_\theta \simeq j\frac{\eta ka}{2} I_0 \frac{e^{-jkr}}{r}\frac{J_1(ka\sin\theta)}{ka\sin\theta}\cos\theta\sin\phi$$

b. $\theta = 0, \phi = \pi/2$

$E_\phi = 0$

$$E_\theta \simeq \frac{j\eta ka}{4} I_0 \frac{e^{-jkr}}{r}$$

$$W_{av} \simeq \frac{|\bar{E}|^2}{2\eta} = \frac{\eta}{32} I_0 \frac{(ka)^2}{r^2}$$

$$U(\theta=0, \phi=\tfrac{\pi}{2}) = \frac{\eta}{32} I_0 (ka)^2$$

5-28. a. From Figure 5.14(a) the smallest changes in reactance for the smallest $h$ ($h<\lambda$) occur when

$$2k_0 h \simeq 2.85 \Rightarrow h = \frac{2.85}{2k_0} = \frac{2.85}{2(2\pi)}\lambda_0 \simeq 0.2268\lambda_0$$

b. For a. $2k_0 h \simeq 2.85 \Rightarrow -\Delta R/R_0 \simeq 0.4$ <from Figure 5.14(a)>

Thus $\Delta R \simeq -0.4R_0$

and $R_r = R_0 + \Delta R = R_0 - 0.4R_0 = 0.6R_0$

According to (5-24)

$$R_0 = 20\pi^2\left(\frac{C}{\lambda_0}\right)^4 = 20\pi^2\left(\frac{2\pi a}{\lambda_0}\right)^4 = 20\pi^2\left(\frac{2\pi}{25}\right)^4 = 0.78757$$

Therefore

$$R_r = 0.6R_0 = 0.6(0.78757) = 0.4725 \text{ ohms}$$

5-29. a. $R_{in} = R_r = 20\pi^2\left(\frac{c}{\lambda}\right)^4 = 20\pi^2\left(\frac{2\pi a}{\lambda}\right)^4 = 20\pi^2\left(\frac{2\pi}{\lambda}\frac{\lambda}{10\pi}\right)^4$

$= 0.3158$, $\lambda = \frac{3\times10^8}{10^9} = 0.3\,m \Rightarrow a = \frac{\lambda}{10\pi} = \frac{0.3}{10\pi} = 0.0095\,m$

$X_{in} \simeq \omega\mu_0 a\left[\ln\left(\frac{8a}{b}\right) - 1.75\right] = 2\pi(10^9)\,4\pi\times10^{-7}\left(\frac{0.3}{10\pi}\right)\left[\ln\left(8\frac{\lambda/10\pi}{\lambda/500\pi}\right) - 1.75\right]$

$= 8\pi(3)\left[\ln(400) - 1.75\right] = 319.798$

$R_{in} = 0.3158$, $X_{in} = 319.798$

b. $2k_0h = \frac{2\pi}{\lambda}(2)(0.06\lambda) = 0.75398$

$N_1 = \sqrt{\frac{\sigma_1 + j\omega\epsilon_1}{j\omega\epsilon_0}} \Rightarrow \sigma_1 + j\omega\epsilon_1 = 10^{-4} + j2\pi\times10^9\times\left(\frac{4}{36\pi}\times10^{-9}\right) = 10^{-4} + j\frac{4}{18}$

$\simeq j\frac{4}{18} = j\omega\epsilon_1$

$N_1 = \sqrt{\frac{\sigma_1 + j\omega\epsilon_1}{j\omega\epsilon_0}} \simeq \sqrt{\frac{j\omega\epsilon_1}{j\omega\epsilon_0}} = \sqrt{\frac{\epsilon_1}{\epsilon_0}} = \sqrt{\epsilon_{r2}} = \sqrt{4} = 2 \Rightarrow N_1^2 = 4.$

Using Figure 5.14(d) $\Rightarrow \frac{\Delta R}{R_0} \simeq 1.0 \Rightarrow \frac{\Delta X}{R_0} \simeq 0.16 \Rightarrow \begin{cases} \Delta R \simeq R_0 = 0.3158 \\ \Delta X \simeq 0.1R_0 = 0.03158 \end{cases}$

c. $R_{in}' = R_{in} + \Delta R = 0.3158\cdot(2) = 0.6316$

$X_{in}' = X_{in} + \Delta X = 319.798 + 0.03158 = 319.82958$

d. $\Gamma_{in} = \frac{Z_{in} - Z_c}{Z_{in} + Z_c} = \frac{0.6316 + j319.82958 - 300}{0.6316 + j319.82958 + 300} = \frac{438.078\angle 133.107^\circ}{438.942\angle 46.772^\circ}$

$= 0.9980\angle 86.335^\circ$

e. $VSWR = \frac{1 + |\Gamma_{in}|}{1 - |\Gamma_{in}|} = \frac{1 + 0.9980}{1 - 0.9980} = 999.$

6-1. a. $E_t = E_1 + E_2 + E_3 = 2E_0 \frac{e^{-jkr}}{r} + E_0 \frac{e^{-jkr_1}}{r_1} + E_0 \frac{e^{-jkr_2}}{r_2}$

where the center element is placed at the origin. For far-field observations

$$\left.\begin{aligned} r_1 &\simeq r - d\cos\theta \\ r_2 &\simeq r + d\cos\theta \end{aligned}\right\} \text{ for phase variations}$$

$r_1 \simeq r_2 \simeq r$ for amplitude variations

and

$$E_t \simeq E_0 \frac{e^{-jkr}}{r}\left\{2 + e^{jkd\cos\theta} + e^{-jkd\cos\theta}\right\}$$

$$\simeq E_0 \frac{e^{-jkr}}{r}\left\{2\left[1 + \frac{1}{2}\left(e^{jkd\cos\theta} + e^{-jkd\cos\theta}\right)\right]\right\}$$

$$= E_0 \frac{e^{-jkr}}{r}\left\{2\left[1 + \cos(kd\cos\theta)\right]\right\}$$

Computer Program
Directivity
$U = \cos^4\left(\frac{\pi}{4}\cos\theta\right)$
At. $d = \lambda/4$
Prad = 8.7119
$D_o = 1.44244$
$= 1.5910\,dB$
$kd = \frac{2\pi}{\lambda}\cdot\frac{\lambda}{4} = \frac{\pi}{2}$
$AF(\theta) = 4\cos^2\left(\frac{\pi}{4}\cos\theta\right)$

Thus the array factor is equal to

$$AF(\theta) = 2\left[1 + \cos(kd\cos\theta)\right] = 4\cos^2\left(\frac{kd}{2}\cos\theta\right)$$

which in normalized form can also be written as

$$AF(\theta)_n = 1 + \cos(kd\cos\theta) = 2\cos^2\left(\frac{kd}{2}\cos\theta\right)$$

b. The nulls of the pattern can be found using either of the above forms for the array factor. For example

| One Form | the other Form |
|---|---|
| $AF(\theta) = 1 + \cos(kd\cos\theta_n) = 0$ | $2\cos^2\left(\frac{kd}{2}\cos\theta_n\right) = 0$ |
| $\cos(kd\cos\theta_n) = -1$ | $\frac{kd}{2}\cos\theta_n = \cos^{-1}(0) = \frac{n\pi}{2}$, $n = \pm1, \pm3, \cdots$ |
| $kd\cos\theta_n = \cos^{-1}(-1) = n\pi$, $n = \pm1, \pm3, \cdots$ | $\theta_n = \cos^{-1}(n\lambda/(2d))$, $n = \pm1, \pm3, \cdots$ |
| $\theta_n = \cos^{-1}(n\lambda/2d)$, $n = \pm1, \pm3, \pm5, \cdots$ | |

which are of identical form. Therefore both forms yield the same results. Thus for $d = \lambda/4$

$$\theta_n = \cos^{-1}\left(\frac{n\lambda}{2d}\right)_{d=\lambda/4} = \cos^{-1}(2n),\ n = \pm1, \pm3, \cdots \Rightarrow \text{No nulls exist.}$$

c. Similarly the maxima of the pattern can be found using either of the two forms for the array factor. For example

(continued)

6-1 (Cont'd) One Form

$AF(\theta) = 1 + \cos(kd\cos\theta_m) = 2$

$\cos(kd\cos\theta_m) = 1$

$kd\cos\theta_m = \cos^{-1}(1) = 2m\pi,\ m = 0, \pm 1, \cdots$

$\theta_m = \cos^{-1}\left(\frac{m\lambda}{d}\right),\ m = 0, \pm 1, \pm 2, \cdots$

Other Form

$AF(\theta) = 2\cos^2\left(\frac{kd}{2}\cos\theta_m\right) = 2$

$\cos\left(\frac{kd}{2}\cos\theta_m\right) = \pm 1$

$\frac{kd}{2}\cos\theta_m = \cos^{-1}(\pm 1) = m\pi,\ m = 0, \pm 1, \cdots$

$\theta_m = \cos^{-1}\left(\frac{m\lambda}{d}\right),\ m = 0, \pm 1, \pm 2, \cdots$

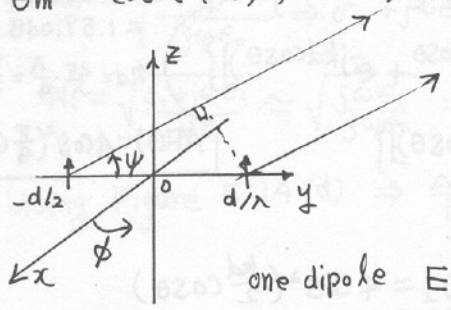
which are of identical form. Therefore both yield the same results.

Thus for $d = \lambda/4$.

$\theta_m = \cos^{-1}(4m),\ m = 0, \pm 1, \pm 2, \rightarrow$

- $m = 0$: $\theta_0 = \cos^{-1}(0) = 90°$
- $m = \pm 1$: $\theta_1 = \cos^{-1}(4) \Rightarrow$ Does not exist.
- The same is true for other values of m (i.e. $m = \pm 2, \pm 3, \cdots$). Therefore the only maxima occur at $\theta = 90°$

6-2.




one dipole $E_\theta \simeq j\eta\frac{kI_0le^{-jkr}}{4\pi r}\cdot\sin\theta$

Array Factor:

$(AF) = E_0\left[e^{j\frac{\pi}{2}}\cdot e^{-j\frac{\lambda}{8}\cdot\frac{2\pi}{\lambda}\cos\psi} + e^{+j\frac{\lambda}{8}\frac{2\pi}{\lambda}\cos\psi}\right]$

$= E_0 e^{j\frac{\pi}{4}}\left[e^{-j\frac{\pi}{4}(\cos\psi - 1)} + e^{j\frac{\pi}{4}(\cos\psi - 1)}\right]$

$= E_0 e^{j\frac{\pi}{4}}\cdot 2\cdot\cos\left(\frac{\pi}{4}(\cos\psi - 1)\right) = E_0 e^{j\frac{\pi}{4}}\cdot 2\cos\left(\frac{\pi}{4}(\sin\theta\sin\phi - 1)\right)$

$\left(\hat{a}_y\cdot\hat{a}_r = \sin\theta\cdot\sin\phi = \cos\psi\right)$ $\overset{=\sin\theta}{}$ At y-z plane, $\phi = 90°$

a. (1) $|E_\theta(\theta)| \propto \left|\sin\theta\cdot\cos\left(\frac{\pi}{4}\right)\right|$, (x-z plane)

$\phi = 0°$;

(2) $|E_\theta(\theta)| \propto \left|\sin\theta\cdot\cos\left(\frac{\pi}{4}(\sin\theta - 1)\right)\right|$, (y-z plane)

($0 < \theta < \pi$; $-\pi < \theta < \pi$)

$\phi = 90°$

(3) $|E_\theta(\phi)| \propto \left|\cos\left(\frac{\pi}{4}(\sin\phi - 1)\right)\right|$, (x-y plane)

$\theta = 90°$

(4) $|E_\phi(\theta)|_{\phi = 0°} \propto 0$

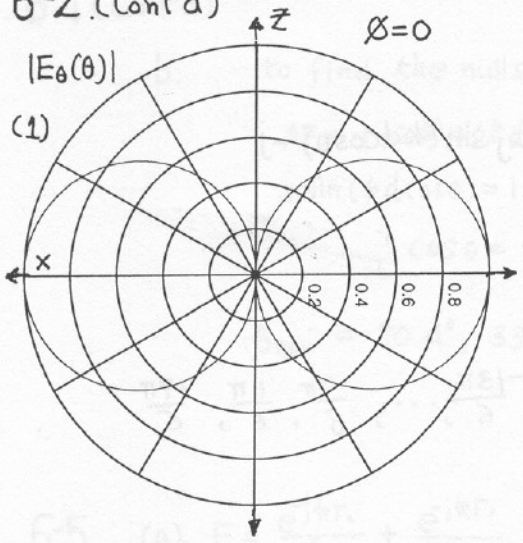
(5) $|E_\phi(\theta)|_{\phi = 90°} \propto 0$

(6) $|E_\phi(\theta)|_{\theta = 90°} \propto 0$

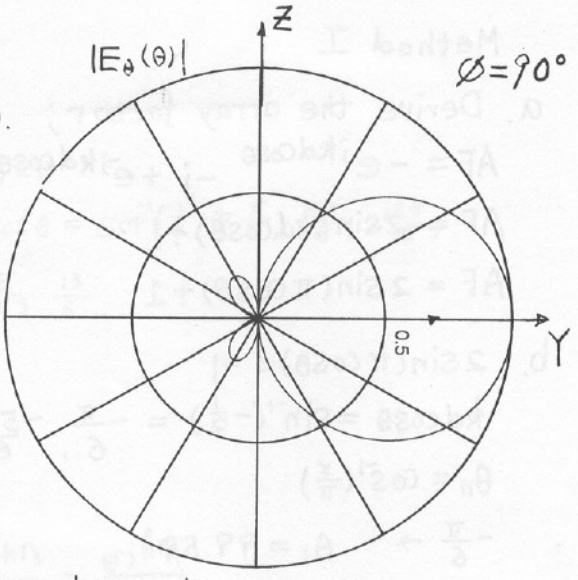
(Continued)

6-2. (Cont'd)

(1) $|E_\theta(\theta)|$ $\phi=0$

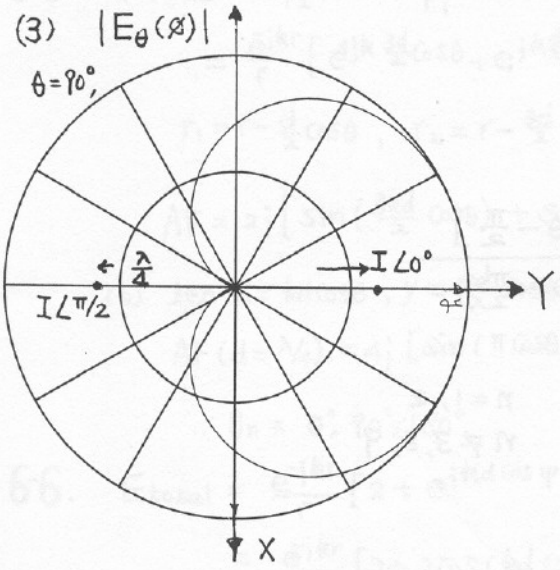



(2). $|E_\theta(\theta)|$ $\phi=90°$




(3) $|E_\theta(\phi)|$

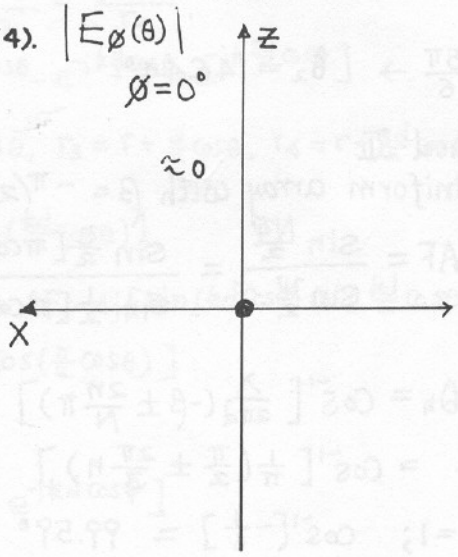
$\theta=90°$,




(4). $|E_\phi(\theta)|$

$\phi=0°$

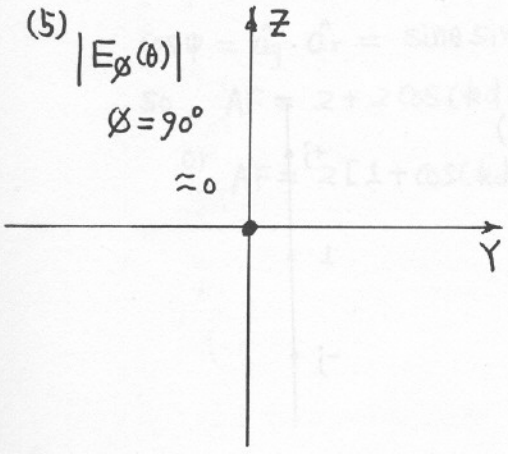
$\approx 0$




(5) $|E_\phi(\theta)|$

$\phi=90°$

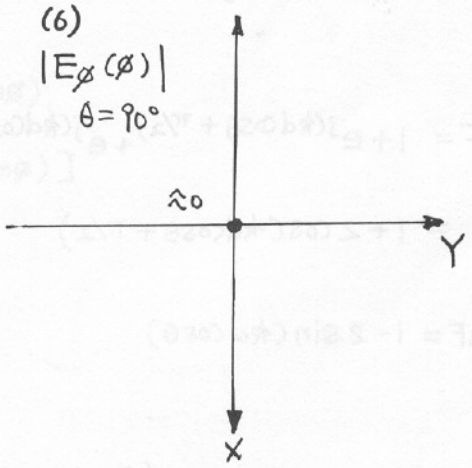
$\approx 0$




(6) $|E_\phi(\phi)|$

$\theta=90°$

$\approx 0$



6-3. Method I

a. Derive the array factor;

$AF = -e^{jkd\cos\theta} - j + e^{-jkd\cos\theta} = -2j\sin(kd\cos\theta) - j$

$AF = 2\sin(kd\cos\theta) + 1$

$AF = 2\sin(\pi\cos\theta) + 1$

b. $2\sin(\pi\cos\theta) = -1$

$kd\cos\theta = \sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}, -\frac{5\pi}{6}, -\frac{13\pi}{6}, \cdots, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}$

$\theta_n = \cos^{-1}(\frac{x}{\pi})$

$-\frac{\pi}{6} \rightarrow \quad \theta_1 = 99.59°$

$-\frac{5\pi}{6} \rightarrow \quad \theta_2 = 146.44°$

Method II

Uniform array with $\beta = -\pi/2$

a. $AF = \frac{\sin\frac{N\psi}{2}}{\sin\frac{\psi}{2}} = \frac{\sin\frac{3}{2}[\pi\cos\theta - \frac{\pi}{2}]}{\sin\frac{1}{2}[\pi\cos\theta - \frac{\pi}{2}]}$

b. $\theta_n = \cos^{-1}[\frac{\lambda}{2\pi d}(-\beta \pm \frac{2n}{N}\pi)]$ $\quad n = 1, 2$

$= \cos^{-1}[\frac{1}{\pi}(\frac{\pi}{2} \pm \frac{2\pi}{3}n)]$ $\quad n \neq 3, 6, 9$

$n = 1;\ \cos^{-1}[-\frac{1}{6}] = 99.59°$

$n = 2;\ \cos^{-1}[-\frac{5}{6}] = 146.44°$

6-4.

a. $AF = 1 + e^{j(kd\cos\theta + \pi/2)} + e^{-j(kd\cos\theta + \pi/2)}$

$= 1 + 2\cos(kd\cos\theta + \pi/2)$

$\therefore AF = 1 - 2\sin(kd\cos\theta)$




(Continue)

6-4 (Cont'd)

b. to find the nulls,

$$AF = 1 - 2\sin(kd\cos\theta) = 0$$

$$2\sin(kd(\cos\theta) = 1, \quad \pi\cos\theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}, \ \frac{5\pi}{6}, \ \frac{13\pi}{6}, \ldots$$

$$\cos\theta = \frac{1}{6}, \ \frac{5}{6}, \ \frac{13}{6}, \ldots$$

$$\theta_{null} = 80.4^\circ, \ 33.6^\circ$$

6-5. (a) $$E = \frac{e^{-jkr_2}}{r_2} + \frac{e^{-jkr_1}}{r_1} - \frac{e^{-jkr_3}}{r_3} - \frac{e^{-jkr_4}}{r_4}$$

$$= \frac{e^{-jkr}}{r}\left[e^{+jk\frac{3d}{2}\cos\theta} + e^{jk\frac{d}{2}\cos\theta} - e^{-jk\frac{d}{2}\cos\theta} - e^{-jk\frac{3d}{2}\cos\theta}\right]$$

$$r_1 = r - \frac{d}{2}\cos\theta, \quad r_2 = r - \frac{3d}{2}\cos\theta, \quad r_3 = r + \frac{d}{2}\cos\theta, \quad r_4 = r + \frac{3d}{2}\cos\theta$$

$$AF = 2j\left[\sin\left(\frac{3kd}{2}\cos\theta\right) + \sin\left(\frac{kd}{2}\cos\theta\right)\right]$$

(b) let $x = kd\cos\theta$, $y = \frac{kd}{2}\cos\theta \Rightarrow AF = 4j[\sin(kd\cos\theta)\cos(\frac{kd}{2}\cos\theta)]$

$$AF(d = \lambda/2) = 4j\left[\sin(\pi\cos\theta)\cos\left(\frac{\pi}{2}\cos\theta\right)\right]$$

$\therefore \theta_n = 0^\circ, 90^\circ, 180^\circ$

6-6. $$E_{total} = \frac{e^{-jkr}}{r}\left[2 + e^{jkd\cos\psi} + e^{-jkd\cos\psi}\right]$$

$$= \frac{e^{-jkr}}{r}\left[2 + 2\cos(kd\cos\psi)\right]$$

$\cos\psi = \hat{a}_y \cdot \hat{a}_r = \sin\theta\sin\phi$

So, $AF = 2 + 2\cos(kd\sin\theta\sin\phi)$

or $AF = 2[1 + \cos(kd\sin\theta\sin\phi)]$

6-7. Placing one element at the origin and the other at d distance above it, the array factor is equal to

$$AF(\theta) = 1 + e^{j(kd\cos\theta+\beta)} = 2e^{j\frac{1}{2}(kd\cos\theta+\beta)}\left[\frac{e^{j\frac{1}{2}(kd\cos\theta+\beta)} + e^{+j\frac{1}{2}(kd\cos\theta+\beta)}}{2}\right]$$

$$AF(\theta) = 2e^{j\frac{1}{2}(kd\cos\theta+\beta)}\cos\left[\tfrac{1}{2}(kd\cos\theta+\beta)\right]$$

which in normalized form can be written as

$$(AF)_n = \cos\left[\tfrac{1}{2}(kd\cos\theta+\beta)\right]$$

a. $\beta = -kd = -\frac{2\pi}{\lambda}\left(\frac{\lambda}{4}\right) = -\frac{\pi}{2}$.

b. For $d = \lambda/4$, $(AF)_n = \cos\left[\frac{\pi}{4}(\cos\theta - 1)\right]$

c. $(AF)_n|_{max} = 1 = \cos\left[\frac{\pi}{4}(\cos\theta_m - 1)\right] \Rightarrow \theta_m = 0°$

$$(AF)_n = 0.707 = \cos\left[\tfrac{\pi}{4}(\cos\theta_h - 1)\right] \Rightarrow \tfrac{\pi}{4}(\cos\theta_h - 1) = \cos^{-1}(0.707) = \begin{cases} +\frac{\pi}{4} \\ -\frac{\pi}{4} \end{cases}$$

For $+\pi/4 \Rightarrow \cos\theta_h - 1 = 1 \Rightarrow \cos\theta_h = 2 \Rightarrow \theta_h = \cos^{-1}(2) \Rightarrow$ Does not exist

For $-\pi/4 \Rightarrow \cos\theta_h - 1 = -1 \Rightarrow \cos\theta_h = 0 = \theta_h = \cos^{-1}(0) = 90° = \frac{\pi}{2}$ radians

Therefore $\theta_{1r} = \theta_{2r} = 2\left(\frac{\pi}{2} - \theta\right) = \pi$

and $D_0 \simeq \frac{4\pi}{\theta_{1r}\theta_{2r}} = \frac{4\pi}{(\pi)^2} = \frac{4}{\pi} = 1.273 = 1.049\text{dB}$

Computer Program $\left(U = \cos^2\left[\frac{\pi}{4}(\cos\theta - 1)\right]\right)$

$$D_0 \simeq 1.42451 = 1.5367\text{ dB}$$

6-8. a. $\beta = +kd = +\frac{\pi}{2}$

b. $(AF)_n = \cos\left[\frac{\pi}{4}(\cos\theta + 1)\right]$

$(AF)_n|_{max} = 1 = \cos\left[\frac{\pi}{4}(\cos\theta_m + 1)\right] \Rightarrow \theta_m = 180° = \pi$ radians

$(AF)_n = 0.707 = \cos\left(\frac{\pi}{4}(\cos\theta_h + 1)\right) \Rightarrow \theta_h = 90° = \frac{\pi}{2}$ radians

$\theta_{1r} = \theta_{2r} = 2(\pi - \frac{\pi}{2}) = \pi$

and

$D_0 \simeq \frac{4\pi}{\pi^2} = \frac{4}{\pi} = 1.273 = 1.049\text{ dB}$

Computer Program result.

$U = \cos^2\left[\frac{\pi}{4}(\cos\theta + 1)\right]$

$D_0 = 1.42451 = 1.5367\text{dB}$

6-9 a. $\beta = -kd = -\frac{2\pi}{\lambda}\left(\frac{\lambda}{2}\right) = -\pi = -180^\circ$

b. $\theta_n = \cos^{-1}\left[1 - \frac{n\lambda}{Nd}\right] = \cos^{-1}\left(1 - \frac{n\lambda}{4\lambda/2}\right) = \cos^{-1}\left(1 - \frac{n}{2}\right)$, $n = 1, 2, 3, \cdots, n \neq 4, 8, \cdots$

$n=1$ : $\theta_1 = \cos^{-1}(1/2) = 60^\circ$

$n=2$ : $\theta_2 = \cos^{-1}(0) = 90^\circ$

$n=3$ : $\theta_3 = \cos^{-1}(-1/2) = 120^\circ$

c. $\theta_m = \cos^{-1}(1 - m\lambda/d) = \cos^{-1}(1 - m\lambda/\lambda/2) = \cos^{-1}(1-2m)$, $m = 0, 1, 2, \cdots$

$m=0$ : $\theta_0 = \cos^{-1}(1) = 0^\circ$

$m=1$ : $\theta_1 = \cos^{-1}(-1) = 180^\circ$

d. $\Theta_0 = 2\cos^{-1}\left(1 - \frac{\lambda}{Nd}\right) = 2\cos^{-1}\left(1 - \frac{\lambda}{4\lambda/2}\right) = 2\cos^{-1}\left(1 - \frac{1}{2}\right) = 2\cos^{-1}\left(\frac{1}{2}\right) = 2(60^\circ)$

$\Theta_0 = 120^\circ$

e. $D_0 = 4N\left(\frac{d}{\lambda}\right) = 4(4)\left(\frac{\lambda/2}{\lambda}\right) = 8 = 9.03\,\text{dB}$

6-10. a. $D_0 = 4N\left(\frac{d}{\lambda}\right)$

$20 = 10\log_{10} D_0\,(\text{dimensionless}) \Rightarrow D_0\,(\text{dimensionless}) = 10^2 = 100$

$100 = 4N\left(\frac{\lambda}{4\lambda}\right) = N \Rightarrow N = 100$

b. $L = 99\left(\frac{\lambda}{4}\right) = \frac{99}{4}\lambda = 24.75\lambda$

c. $\Theta_{3dB} = \Theta_h = 2\cos^{-1}\left(1 - \frac{1.391\lambda}{Nd\pi}\right)\Big|_{n=100} = 2\cos^{-1}\left(1 - \frac{1.391\lambda}{\pi(\frac{\lambda}{4})100}\right)$

$= 2\cos^{-1}\left(1 - \frac{1.391(4)}{100\pi}\right) = 2\cos^{-1}(1 - 0.01771) = 2\cos^{-1}(0.98228)$

$\Theta_h = 2(10.799^\circ) = 21.598^\circ \simeq 21.6^\circ$

d. Sidelobe (dB) $\simeq -13.5\,\text{dB}$

e. $\beta = \pm kd = \pm\frac{2\pi}{\lambda}\left(\frac{\lambda}{4}\right) = \pm\frac{\pi}{2} = \pm 90^\circ$

6-11. a. Choose different phase excitation. That is

$\beta = \pm\left(kd + \frac{2.94}{N}\right) \simeq \pm\left(kd + \frac{\pi}{N}\right)$

$\beta = \pm\left(\frac{2\pi}{\lambda}\frac{\lambda}{4} + \frac{2.94}{100}\right) = \pm\left(\frac{\pi}{2} + 0.0294\right) = \pm(1.570796 + 0.0294) = \pm(1.6) = \pm 91.684^\circ$

b. Directivity increase by 1.1789 factor $= 2.526\,\text{dB}$

c. The HPBW will decrease because sidelobe level will increase.

$\Theta_h = 2\cos^{-1}\left(1 - \frac{0.1398\lambda}{Nd}\right) = 2\cos^{-1}\left(1 - \frac{0.1398\lambda}{100\,\lambda/4}\right) = 2\cos^{-1}\left(1 - \frac{0.1398(4)}{100}\right)$

$= 2\cos^{-1}(1 - 0.005592) = 2\cos^{-1}(0.9944) = 2(6.066^\circ) = 12.13^\circ$

decreased by $9.47^\circ$

d. sidelobe level will increase. It will be higher than $-13.5\,\text{dB}$

6-12. a. $d = \left(\frac{N-1}{N}\right)\frac{\lambda}{4} = 0.225\lambda$

b. $\beta = kd + \frac{2.94}{10} = 2\pi(0.225) + 0.294 = 1.7077 \text{ rad}$

c. $\theta_n = \cos^{-1}\left(1 + (1-2n)\frac{\lambda}{2dN}\right)$

$\theta_n = \cos^{-1}\left(1 + (1-2n)\frac{1}{4.5}\right)$

$\theta_1 = \cos^{-1}(0.777) = 38.9°,\ \theta_2 = \cos^{-1}(0.333) = 70.53°,$

$\theta_3 = \cos^{-1}(-0.111) = 96.38°,\ \theta_4 = \cos^{-1}(-0.555) = 123.7°\ \ldots$

d. First null Beamwidth

$\theta_n = 2\cos^{-1}\left(1 - \frac{\lambda}{2dN}\right) = 2\cos^{-1}\left(1 - \frac{1}{2(0.225)\cdot 10}\right) = 77.88°$

e. $D_o = 1.789\left[4N\cdot\left(\frac{d}{\lambda}\right)\right] = 1.789\left[4\cdot 10\cdot(0.225)\right] = 16.101 = 12.068\text{dB}$

6-13. $N=10,\ d = \lambda/4$

a. Broadside (Table 6.1 and 6.2) $\Rightarrow \beta = 0$

$HPBW = 2\left[90° - \cos^{-1}\left(\frac{1.394\times 4}{10\pi}\right)\right] = 2(90° - 79.80°) = 20.4°$

$FNBW = 2\left[90° - \cos^{-1}\left(\frac{4}{10}\right)\right] = 2(90° - 66.42°) = 47.16°$

$FSLBW = 2\left[90° - \cos^{-1}\left(\frac{6}{10}\right)\right] = 2(90° - 53.13°) = 73.74°$

From (6-17a) $\Rightarrow$ Relative sidelobe maximum $= -13.46\text{dB}$

From Table 6.7 $\Rightarrow D_o = 2N\left(\frac{d}{\lambda}\right) = 2\cdot 10\cdot\frac{1}{4} = 5 = 6.99\text{dB}$

Using the computer program at the end of Chapter 2

$D_o = 5.21 \Rightarrow 7.17\text{dB}$

(Continued)

6-13 (Cont'd)

b. Ordinary End-Fire (Tables 6.3 and 6.4) $\Rightarrow \beta = \pm kd = \pm \pi/2 = \pm 90°$

$HPBW = 2\cos^{-1}\left[1 - \frac{1.391(4)}{10\pi}\right] = 2(34.62°) = 69.25°$

$FNBW = 2\cos^{-1}\left[1 - \frac{4}{10}\right] = 2\cos^{-1}(0.6) = 2(53.13°) = 106.26°$

$FSLBW = 2\cos^{-1}\left[1 - \frac{3(4)}{20}\right] = 2(66.42) = 132.84°$

From (6-19a) $\Rightarrow$ Relative side lobe maximum $= -13.46$ dB

From Table 6.7 $\Rightarrow D_o = 4N\left(\frac{d}{\lambda}\right) = 4(10)\frac{1}{4} = 10 = 10$ dB

Using the computer program at the end of Chapter 2 $\Rightarrow D_o = 10.05 = 10.02$ dB

c. Hansen-Woodyard End-Fire (Tables 6.5 and 6.6)

$\beta = \pm\left(kd + \frac{\pi}{10}\right) = \pm(90° + 18°) = \pm 108°$

$HPBW = 2\cos^{-1}\left[1 - \frac{1.398(4)}{10}\right] = 2(19.25) = 38.5°$

$FNBW = 2\cos^{-1}\left[1 - \frac{4}{2(10)}\right] = 2(36.87) = 73.74°$

$FSLBW = 2\cos^{-1}\left[1 - \frac{4}{10}\right] = 2(53.13) = 106.26°$

From Figure 6.9 $\Rightarrow$ Relative side lobe maximum $\simeq -9$ dB

From Table 6.7 $\Rightarrow D_o = 1.789[4N(\frac{d}{\lambda})] = 1.789(4)(10)(\frac{1}{4}) = 17.89 = 12.5$ dB

Using the computer program at the end of Chapter 2 $\Rightarrow D_o = 18.02 = 12.56$ dB

6-14. $(AF)_n = \dfrac{\sin\left[\frac{N}{2}(kd\cos\theta + \beta)\right]}{N\sin\left[\frac{1}{2}(kd\cos\theta + \beta)\right]} = \dfrac{\sin\left[5(\frac{\pi}{2}\cos\theta + \beta)\right]}{10\sin\left[\frac{1}{2}(\frac{\pi}{2}\cos\theta + \beta)\right]}$

$\theta_o = 45° \Rightarrow \beta = -kd\cos\theta_o = -\frac{2\pi}{\lambda}\left(\frac{\lambda}{4}\right)\cos 45° = -1.1107$ radians

a. Using (6-22)

$\Theta_h = HPBW = \cos^{-1}\left[\cos 45° - 0.443\frac{\lambda}{(L+d)}\right] - \cos^{-1}\left[\cos 45° + 0.443\frac{\lambda}{(L+d)}\right]$

$= \cos^{-1}\left[0.707 - 0.443\frac{1}{2.25+0.25}\right] - \cos^{-1}\left[0.707 + 0.443\frac{1}{2.25+0.25}\right]$

$\Theta_h = \cos^{-1}(0.5299) - \cos^{-1}(0.8843) = 58° - 27.83° = 30.2°$

b. $D_o = \dfrac{U_{max}}{U_o}$, $U_{max} = 1$

$U_o = \frac{1}{2}\int_0^{\pi}\left[\dfrac{\sin\left[5\cdot\frac{\pi}{2}(\cos\theta - 0.707)\right]}{5\cdot\frac{\pi}{2}(\cos\theta - 0.707)}\right]^2 \sin\theta\, d\theta$

let $z = 5\cdot\frac{\pi}{2}(\cos\theta - 0.707)$, $dz = -\frac{5\pi}{2}\sin\theta\, d\theta$

$U_o = -\frac{1}{5\pi}\int_{2.3012}^{-13.4067}\left[\frac{\sin z}{z}\right]^2 dz$,

(N kd → large)

$U_o \simeq -\frac{1}{5\pi}\int_{\infty}^{-\infty}\left(\frac{\sin z}{z}\right)^2 dz = \frac{\pi}{5\pi} = \frac{1}{5}$

(Continued)

6-14 (Cont'd)

$$D_o = \frac{U_{max}}{U_o} \simeq \frac{1}{(1/5)} = 5 = D_o\Big|_{\theta_o=90^\circ}$$

Since for $\theta_o = 90^\circ$

$D_o \simeq 2N\left(\frac{d}{\lambda}\right)$ and for $\theta_o = 0^\circ$ $D_o \simeq 4N\left(\frac{d}{\lambda}\right)$

We might expect for $\theta_o = 45^\circ$ the value of $D_o$ to be somewhere between $D_o\Big|_{\theta_o=0^\circ}$ and $D_o\Big|_{\theta_o=90^\circ}$

A possibility is $D_o\Big|_{\theta_o=45^\circ} \simeq 3N\left(\frac{d}{\lambda}\right) = 3\cdot(10)\cdot\left(\frac{1}{4}\right)$, $D_o \simeq 7.5$

Using computer Program:

$$D_o = 5.3221$$

6-15. $$(AF)_n = \frac{\sin\left[\frac{N}{2}(kd\cos\theta+\beta)\right]}{N\sin\left[\frac{1}{2}(kd\cos\theta+\beta)\right]}$$

a. For $\beta = 0 \Rightarrow (AF)_n = \dfrac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{N\sin\left(\frac{1}{2}kd\cos\theta\right)}$

In order for the array not to have any minor lobes, we can assume that its first null occurs at $\theta = 0^\circ$ or $180^\circ$. Thus

$$(AF)_n = \frac{\sin\left(\frac{N}{2}kd\right)}{N\sin\left(\frac{1}{2}kd\right)} = 0 \Rightarrow \frac{N}{2}kd = \pi \Rightarrow d = \frac{2\pi}{kN} = \frac{\lambda}{N}$$

This assures that there are no minor lobes formed.

b. For $\beta = kd$ the maximum occurs at $\theta = 180^\circ$ and the array factor can be written as $(AF)_n = \dfrac{\sin\left[\frac{N}{2}kd(\cos\theta+1)\right]}{N\sin\left[\frac{1}{2}kd(\cos\theta+1)\right]}$

In order for the array not to have any minor lobes, we can assume that the first null is formed at $\theta = 0^\circ$.

Thus $\frac{N}{2}kd(\cos\theta+1)\Big|_{\theta=0^\circ} = Nkd = \pi \Rightarrow d = \frac{\pi}{Nk} = \frac{\lambda}{2N}$

6-16. $kd = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$

a. $\beta = 0$ radians

b. $\beta = -\pi/2$

c. $\beta = +\pi/2$

d. $\beta = -1.36 = -\frac{\sqrt{3}}{4}\pi = -0.433\pi$

e. $\beta = -(\frac{\pi}{2} + 0.147)$ or $-(\frac{\pi}{2} + 0.157) = -\frac{11}{20}\pi = -1.72$

f. $\beta = +(\frac{\pi}{2} + 0.147)$ or $+(\frac{\pi}{2} + 0.157) = \frac{11}{20}\pi = 1.72$

6-17. $N = 19,\ d = \lambda/4$

a. $\beta = -kd\cos\theta_0\Big|_{\substack{\theta_0=30^\circ \\ d=\lambda/4}} = -\frac{2\pi}{\lambda}(\frac{\lambda}{4})\cos(30^\circ) = -\frac{\pi}{2}\frac{\sqrt{3}}{2} = -\frac{\pi\sqrt{3}}{4} = -1.3603$

$\beta = -\frac{\pi\sqrt{3}}{4} = -1.3603\ (\text{rad}) = -77.942^\circ$

b. $\theta_h = \cos^{-1}\left[\cos\theta_0 - 0.443\frac{\lambda}{L+d}\right]_{\theta_0=30^\circ} - \cos^{-1}\left[\cos\theta_0 + 0.443\frac{\lambda}{L+d}\right]_{\theta_0=30^\circ}$

$= \cos^{-1}\left[0.866 - \frac{0.443}{5}\right] - \cos^{-1}\left[0.866 + \frac{0.443}{5}\right]$

$= \cos^{-1}(0.7774) - \cos^{-1}(0.9546) = 38.9769^\circ - 17.3309^\circ = 21.6459^\circ$

$\theta_h = 21.6459^\circ$

C. **13.5 dB**

Computer Result.

HPBW $\Rightarrow$ 23 degree.

$D_0$ (Directivity) = 10.103 dB

6-18. $D_o \simeq 2N(d/\lambda)$

a. $d = \frac{\lambda}{4}$, $D_o \simeq 2 \cdot 10 \cdot \frac{1}{4} = 5 = 6.99\,dB$

Computer Program : $D_o = 7.132\,dB$

b. $d = \frac{\lambda}{2}$, $D_o \simeq 2 \cdot 10 \cdot \frac{1}{2} = 10 = 10\,dB$

Computer Program : $D_o = 10.00\,dB$

c. $d = \frac{3\lambda}{4}$, $D_o \simeq 2 \cdot 10 \cdot (0.75) = 15 = 11.76\,dB$

Computer Program : $D_o = 11.624\,dB$

d. $d = \lambda$, $D_o \simeq 2 \cdot 10 \cdot (1) = 20 = 13.0\,dB$

Computer Program : $D_o = 10.011\,dB$

6-19. The recommended element spacing is

$d = \frac{1}{1+\cos\theta}$, where $\theta$ is the scan angle in degrees

a. $\theta_o = 30°$

$d = \frac{1}{1+\cos 30°} = 0.5359$ wavelength

b. $\theta_o = 45°$

$d = \frac{1}{1+\cos 45°} = \frac{1}{1+0.7071} = 0.58578$ wavelength

c. $\theta_o = 60°$

$d = \frac{1}{1+\cos 60°} = \frac{1}{1+0.5} = 0.6667$ wavelength.

Although a narrow element pattern can sometimes accommodate larger spacing, using this rule will ensure that the array factor has only one maximum in the visible region

6-20. Since the excitation coefficient of each element is identical, $\beta = 0$, Thus $AF = e^{j\psi_o} + e^{j\psi_x} + e^{j\psi_y} + e^{j\psi_z}$

where

$\psi_o = 0$ ; For element at origin

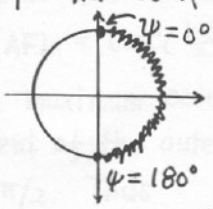
$\psi_x = kd\cos\gamma_x = kd\,\hat{a}_x \cdot \hat{a}_r = kd\sin\theta\cos\phi$ ; For element along x-axis

$\psi_y = kd\cos\gamma_y = kd\,\hat{a}_y \cdot \hat{a}_r = kd\sin\theta\cos\phi$ ; For element along Y-axis

$\psi_z = kd\cos\gamma_z = kd\,\hat{a}_z \cdot \hat{a}_r = kd\cos\theta$ ; For element along Z-axis

6-21. $\beta = 0°$, $d = \lambda/4$

a. $\psi = kd\cos\theta + \beta = \frac{2\pi}{\lambda}\frac{\lambda}{4}\cos\theta + 0 = \frac{\pi}{2}\cos\theta$




b. $\theta = 0° \Rightarrow \psi_1 = \frac{\pi}{2}\cos 0° = \frac{\pi}{2}\cos 0° = \frac{\pi}{2} \Rightarrow z_1 = j$

$\theta = 45° \Rightarrow \psi_2 = \frac{\pi}{2}\cos 45° = \frac{\pi}{2}\frac{1}{\sqrt{2}} \Rightarrow z_2 = 0.444 + j0.896$

$AF = (z - j)(z - 0.444 - j0.896) = (-0.896 + j0.444) + z(-0.444 - j1.896) + z^2$

3 elements needed

c. $a_1 = -0.896 + j0.444$

$a_2 = -0.444 - j1.896$

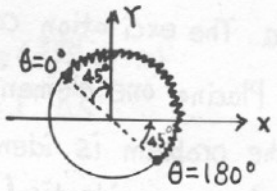
$a_3 = 1$

6-22. $\beta = \pi/4$, $d = \lambda/4$

a. $\psi = kd\cos\theta + \beta = \frac{2\pi}{\lambda}\cdot\frac{\lambda}{4}\cos\theta + \frac{\pi}{4} = \frac{\pi}{2}\cos\theta + \frac{\pi}{4}$

visible region: $\theta = 0° \Rightarrow \psi = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$

$\theta = 180° \Rightarrow \psi = -\frac{\pi}{2} + \frac{\pi}{4} = -\frac{\pi}{4}$




b. $AF = (z - z_1)(z - z_2)(z - z_3)$

$= a_1 + a_2 z + a_3 z^2 + a_4 z^3$ → 4 element required

c. $\psi_{100} = kd\cos\theta + \beta = 90\cdot\cos(10°) + 45° = 133.633° = -0.690 + j0.724 = z_1$

$\psi_{70°} = 90°\cos(70°) + 45° = 75.782° = 0.2456 + j0.9694 = z_2$

$\psi_{110°} = 90°\cos(110°) + 45° = 14.218° = 0.9694 + j0.2456 = z_3$

So $AF = (z + 0.690 - j0.724)(z - 0.2456 - j0.9694)(z - 0.9694 - j0.2456)$

$= [z^2 + (0.4444 - j1.6934)z + (-0.8713 - j0.491)](z - 0.9694 - j0.2456)$

$AF = z^3 + z^2(-0.5250 - j1.9390) + z(-1.718 + j1.041) + (0.724 + j0.690)$

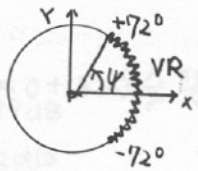
So $a_1 = +0.724 + j0.690 = 1.000138 \angle 43.62°$

$a_2 = -1.718 + j1.041 = 2.00898 \angle 148.786°$

$a_3 = -0.5250 - j1.9390 = 2.0088 \angle -105.1500°$

$a_4 = 1$

6-23. a. $\psi = kd\cos\theta + \beta = 72^\circ\cos\theta$




b. $\theta = 0^\circ: \psi = \psi_1 = 0.4\pi \Rightarrow z_1 = 0.31 + j0.95$

$\theta = 50^\circ: \psi = \psi_2 = 0.257\pi \Rightarrow z_2 = 0.69 + j0.723$

$\theta = 100^\circ: \psi = \psi_3 = -0.218 \Rightarrow z_3 = 0.976 - j0.216$

$AF = (z - 0.31 - j0.95)(z - 0.69 - j0.723)(z - 0.976 + j0.216)$

$AF = z^3 + z^2(-1.98 - j1.46) + z(0.865 + j2.298) + (0.272 - j0.96)$

4 elements required

C. $a_1 = 0.272 - j0.962 = 1\angle -74.22^\circ$

$a_2 = 0.865 + j2.298 = 2.46\angle 69.37^\circ$

$a_3 = -1.98 - j1.46 = 2.46\angle 216.4^\circ$

$a_4 = 1\angle 0^\circ = 1 + j0$

6-24. a. The excitation coefficients for a 3-element array are 1, 2, 1.

Placing one element at the origin, one above it, and the other below it, the problem is identical to that of Problem 6.1. Thus the array factors are identical and equal to

b. $(AF)_n = 1 + \cos(kd\cos\theta) = 2\cos^2\left(\frac{kd}{2}\cos\theta\right)$

C. The nulls of the pattern can be found using either of the above forms, as it was demonstrated in Problem 6.1. Using either one

$d = \lambda \Rightarrow \theta_n = \cos^{-1}(n\lambda/2d) = \cos^{-1}(n/2)$, $n = \pm 1, \pm 3, \pm 5, \cdots$

$n = \pm 1: \theta_1 = \cos^{-1}(\pm 1/2) = \cos^{-1}(\pm 0.5) = 60^\circ, 120^\circ$

$n = \pm 3: \theta_3 = \cos^{-1}(\pm 3/2) = \cos^{-1}(\pm 1.5) =$ Does not exist.

$n = \pm 5: \theta_5 = \cos^{-1}(\pm 5/2) = \cos^{-1}(2.5) =$ Does not exist. The same holds for $|n| \geq 7$.

d. The maxima of the pattern can also be found either of the forms. Using the results of Problem 6.1.

$d = \lambda \Rightarrow \theta_m = \cos^{-1}(m\lambda/d) = \cos^{-1}(m)$, $m = 0, \pm 1, \pm 2, \pm 3, \cdots$

$m = 0: \theta_0 = \cos^{-1}(0) = 90^\circ$

$m = \pm 1: \theta_1 = \cos^{-1}(\pm 1) = 0^\circ, 180^\circ$

$m = \pm 2: \theta_2 = \cos^{-1}(\pm 2) =$ Does not exist. The same holds for $n \geq 3$.

6-25. For a three-element binomial array the array factor is that given in Problem 6.1 and 6.24. Thus in normalized form it can be written as

$$(AF)_n = \cos^2\left(\frac{kd}{2}\cos\theta\right)$$

whose maximum occurs at $\theta = 90°$. In order not to have a side lobe, the argument of the outer cosine function at $\theta = 0°$ or $180°$ must be equal or less than $\pi/2$. Thus

$$\left|\frac{kd}{2}\cos\theta\right|_{\substack{\theta=0°\\ \theta=180°}} \leq \frac{\pi}{2} \Rightarrow d \leq \frac{\pi}{k} = \frac{\lambda}{2}$$

6-26. The excitation coefficients of a 4-element binomial array are 1,3,3,1 or

a. $\left.\begin{matrix} a_1 = 3 \\ a_2 = 1 \end{matrix}\right\} N = 2M = 4 \Rightarrow M = 2$

b. $(AF)_4 = \sum_{n=1}^{M=2} a_n \cos[(2n-1)u]$, $u = \frac{\pi d}{\lambda}\cos\theta$, using (6-61a) and (6-61c).

Thus

$$(AF)_4 = a_1\cos(u) + a_2\cos(3u) = 3\cos\left(\frac{\pi d}{\lambda}\cos\theta\right) + \cos\left(\frac{3\pi d}{\lambda}\cos\theta\right)$$

which can also be written, using (6-66) for $m = 3$, as

$$(AF)_4 = 3\cos\left(\frac{\pi d}{\lambda}\right) + 4\cos^3\left(\frac{\pi d}{\lambda}\cos\theta\right) - 3\cos\left(\frac{\pi d}{\lambda}\cos\theta\right) = 4\cos^3\left(\frac{\pi d}{\lambda}\cos\theta\right)$$

$$(AF)_4 = 4\cos^3\left(\frac{\pi d}{\lambda}\cos\theta\right)$$

c. The nulls occur when

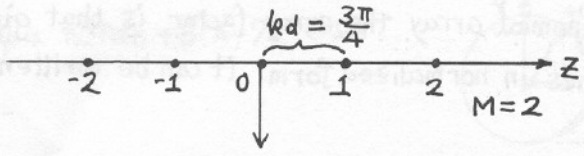
$$(AF)_4 = 4\cos^3\left(\frac{\pi d}{\lambda}\cos\theta_n\right) = 0 \Rightarrow \frac{\pi d}{\lambda}\cos\theta_n = \cos^{-1}(0) = \pm\frac{(2n+1)\pi}{2},\ n = 0, 1, 2, \cdots$$

or $\theta_n = \cos^{-1}\left[\pm\frac{(2n+1)\lambda}{2d}\right] \overset{d=3\lambda/4}{=} \cos^{-1}\left[\pm\frac{(2n+1)2}{3}\right]$, $n = 0, 1, 2, \cdots$

$n = 0$ : $\theta_0 = \cos^{-1}(\pm\frac{2}{3}) = 48.19°,\ 131.81°$

$n = 1$ : $\theta_1 = \cos^{-1}(\pm 2) =$ Does not exist. The same holds for $n \geq 2$.

6-27.




a. Use Pascal's triangle to find excitation coefficients

$2a_0 = 6 \rightarrow a_3 = 3$

$a_1 = 4$

$a_2 = 1$

b. $AF = 2\sum_{n=0}^{2} a_n \cos(n kd\cos\theta)$

$= 2\{3 + 4\cos(kd\cos\theta) + \cos(2kd\cos\theta)\}$

$=$ $4\{1 + 2\cos(kd\cos\theta) + \cos^2(kd\cos\theta)\}$ ← $\cos(2kd\cos\theta) = 2\cos^2(kd\cos\theta) - 1$

$= 4\{1 + \cos(kd\cos\theta)\}^2 = 16\cos^4(\frac{kd}{2}\cos\theta)$

c. $U(\theta,\phi) = |AF|^2 = 256\cos^8(\frac{kd}{2}\cos\theta)$

$U_{max} = U(\theta = \frac{\pi}{2}, \phi) = 256$

$P_n(\theta,\phi) = \frac{U(\theta,\phi)}{U_{max}} = \cos^8(\frac{kd}{2}\cos\theta)$

d. nulls occurs when $\cos(\frac{kd}{2}\cos\theta_n) = 0$

$kd\cos\theta_n = (2n+1)\pi$ , $n = 0, \pm 1, \pm 2, \cdots$

$\theta_n = \cos^{-1}\{(2n+1)\frac{\pi}{kd}\} = \cos^{-1}\{(2n+1)\frac{4}{5}\}$

$n=0$: $\theta_0 = \cos^{-1}(\frac{4}{5}) = 36.9°$

$n=-1$: $\theta_0 = \cos^{-1}(-\frac{4}{5}) = 143.1°$

6-28. The excitation coefficients for a 4-element binomial array are 1,3,3,1 or

a. $a_1 = 3$, $a_2 = 1$

b. Since the elements are placed along the x-axis

$\cos\gamma = \hat{a}_x \cdot \hat{a}_r = \hat{a}_x \cdot (\hat{a}_x \sin\theta\cos\phi + \hat{a}_y \sin\theta\sin\phi + \hat{a}_z\cos\theta) = \sin\theta\cos\phi$

The array factor for this array is similar to that of Problem 6.26. The

$(AF)_4 = 3\cos(\frac{\pi d}{\lambda}\sin\theta\cos\phi) + \cos(\frac{3\pi d}{\lambda}\sin\theta\cos\phi) = 4\cos^3(\frac{\pi d}{\lambda}\sin\theta\cos\phi)$

c. The total field is obtained using the pattern multiplication rule of (6-5) by multiplying the field of a single $\lambda/2$ dipole, as given by (4-84), with the array factor above. Thus

$$E_\theta(\text{total}) = E_\theta(\text{single}) \times (AF) = j\eta\frac{I_0 e^{-jkr}}{2\pi r} \cdot \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\left[4\cos^3(\frac{\pi d}{\lambda}\sin\theta\cos\phi)\right]$$

6-29. The answers to this problem are identical to those of Problem 6.28, except that $\cos\gamma$ is equal to

$$\cos\gamma = \hat{a}_y \cdot \hat{a}_r = \sin\theta\sin\phi$$

Therefore $\sin\theta\cos\phi$ in Prob 6.28 must be replaced by $\sin\theta\sin\phi$

6-30. a. From (6-63), $a_1 = 10,\ a_2 = 5,\ a_3 = 1$, ← Verified with computer program

b. Since the array is broadside, the progressive phase shift between the elements as required by (6-18a) is zero ($\beta = 0$.)

C. $(AF)_6 = 2\sum_{n=1}^{3} a_n \cos[(2n-1)u]$, $u = \frac{\pi d}{\lambda}\cos\theta = \frac{\pi}{2}\cos\theta$, Computer Program ⇒ $D_o = 6.089$ dB At $d = \lambda/2$

d. $\underline{E} = \hat{a}_\theta\, j\eta \frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos(\frac{\pi}{4}\cos\theta) - \cos(\frac{\pi}{4})}{\sin\theta}\right]\left\{10\cos(\frac{\pi}{2}\cos\theta) + 5\cos(\frac{3\pi}{2}\cos\theta) + \cos(\frac{5\pi}{2}\cos\theta)\right\}$

6-31. a. From (6-63), $a_1 = 10,\ a_2 = 15,\ a_3 = 6,\ a_4 = 1$ ← Verified with computer program

b. Same answer like (b) in Problem 6-30. ($D_o = 6.467$ dB) At $d = \frac{\lambda}{2}$

C. $AF = \sum_{n=1}^{4} a_n \cos[2(n-1)u] = 10 + 15\cos 2u + 6\cos 4u + \cos 6u$

($\leftarrow u = \frac{\pi d}{\lambda}\cos\theta = \frac{\pi}{2}\cos\theta$)

d. Field of $E_\theta$ at origin: From (4-62a)

$$E_\theta \simeq j\frac{\eta I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos(\frac{\pi}{4}\cos\theta) - \cos(\frac{\pi}{4})}{\sin\theta}\right] \leftarrow \text{one dipole of } \tfrac{\lambda}{4} \text{ length}$$

Array.

$$E_\theta \simeq j\eta\frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos(\frac{\pi}{4}\cos\theta) - 0.707}{\sin\theta}\right]\left[10 + 15\cos 2u + 6\cos 4u + \cos 6u\right]$$

($\leftarrow u = \frac{\pi d}{\lambda}\cos\theta$)

6-32. The excitation coefficients for a 5-element binomial array are 1, 4, 6, 4, 1 or $a_1 = 3,\ a_2 = 4$, and $a_3 = 1$. Thus the array factor can be written using (6-61b) and (6-61c) as

a. $(AF)_5 = \sum_{n=1}^{3} a_n \cos[2(n-1)u] = a_1 + a_2\cos(2u) + a_3\cos(4u)$

Using (6-69) for $m = 2$ and $m = 4$, the array factor can also be written as $(AF)_5 = a_1 + a_2[2\cos^2 u - 1] + a_3[8\cos^4 u - 8\cos^2 u + 1]$

$(AF)_5 = 3 + 4(2\cos^2 u - 1) + (8\cos^4 u - 8\cos^2 u + 1) = 8\cos^4 u = 8\cos^4(\frac{\pi d}{\lambda}\cos\theta)$

(Continued)

6-32 (Cont'd)

b. Using the computer program at the end of Chapter 2.

$$D_o = 3.668 = 5.64 \text{ dB}$$

c. The nulls of the pattern are obtained from

$$(AF)_5 = 8\cos^4\left(\frac{\pi d}{\lambda}\cos\theta_n\right)\Big|_{d=\lambda} = 8\cos^4(\pi\cos\theta_n) = 0$$

$$\pi\cos\theta_n = \cos^{-1}(0) = \pm\left(\frac{2n+1}{2}\right)\pi, \quad n = 0, 1, 2, 3, 4, \cdots$$

$$\theta_n = \cos^{-1}\left[\pm\left(\frac{2n+1}{2}\right)\right], \quad n = 0, 1, 2, \cdots$$

$n=0$ : $\theta_0 = \cos^{-1}(\pm\frac{1}{2}) = 60°, 120°$

$n=1$ : $\theta_1 = \cos^{-1}(\pm\frac{3}{2})$ = Does not exist. The same holds for $n \geq 2$.

6-33. $R_o = 20$ dB $\Rightarrow R_o$ (voltage Ratio) $= 10^1 = 10$

$$z_o = \frac{1}{2}\left[(10+\sqrt{10^2-1})^{1/2} + (10-\sqrt{10^2-1})^{1/2}\right] = 2.3452$$

a. The array factor can be written as

$$(AF)_3 = \sum_{n=1}^{2} a_n \cos[2(n-1)u] = a_1 + a_2\cos(2u) = a_1 + a_2[2\cos^2 u - 1]$$

$$(AF)_3 = (a_1 - a_2) + 2a_2\cos^2 u = (a_1 - a_2) + 2a_2\cos^2\left(\frac{\pi d}{\lambda}\cos\theta\right)$$

letting $\cos u = z/z_o$, and equating the array factor to the Tschebyscheff polynomial of order 2, we obtain

$$(a_1 - a_2) + 2a_2\left(\frac{z}{z_o}\right)^2 = -1 + 2z^2 \Rightarrow \frac{2a_2}{z_o^2} = 2 \Rightarrow a_2 = z_o^2 = (2.3452)^2 = 5.5$$

Therefore $a_1 = 4.5$, $a_2 = 5.5$ } or normalized { $a_{1n} = 4.5/5.5 = 0.818$, $a_{2n} = 5.5/5.5 = 1.0$

(* $(a_1 - a_2) = -1 \Rightarrow a_1 = a_2 - 1 = 4.5$ )

b. Thus the array factor is equal to

$$(AF)_3 = 0.818 + \cos\left(\frac{2\pi d}{\lambda}\cos\theta\right) = -0.182 + 2\cos^2\left(\frac{\pi d}{\lambda}\cos\theta\right)$$

c. $(AF)_3 = 0.818 + \cos(2\pi\cos\theta_n) = 0 \Rightarrow \cos(2\pi\cos\theta_n) = -0.818$

$2\pi\cos\theta_n = \cos^{-1}(-0.818) =$ { $\pm 144.885° = \pm 2.5287$ rad. Also $(\pm 2.5287 \pm 2\pi n)$ rad.
{ $\pm 215.115° = \pm 3.754$ rad. Also $(\pm 3.754 \pm 2\pi n)$ rad.

Therefore $\theta_n = \cos^{-1}\left(\pm\frac{2.5287}{2\pi}\right) = 66.27°, 113.73°$

$$\theta_n = \cos^{-1}\left(+\frac{2.5287 - 2\pi}{2\pi}\right) = 126.69°$$

$$\theta_n = \cos^{-1}\left(\frac{-2.5287 + 2\pi}{2\pi}\right) = 53.31°$$

(Continued)

6-33. (Cont'd)

$$\theta_n = \cos^{-1}\left(\pm\frac{3.754}{2\pi}\right) = 53.31°, 126.69°$$

$$\theta_n = \cos^{-1}\left(\frac{+3.754-2\pi}{2\pi}\right) = 113.73°$$

$$\theta_n = \cos^{-1}\left(\frac{-3.754+2\pi}{2\pi}\right) = 66.27°$$

Therefore the nulls occur at

$\theta_n = 53.31°$, $66.27°$, $113.73°$, and $126.69°$

d. The maxima of the pattern are obtained using

$$(AF)_3\Big|_{max} = 1.818 = 0.818 + \cos(2\pi\cos\theta_m) \Rightarrow \cos(2\pi\cos\theta_m) = 1$$

$$2\pi\cos\theta_m = \cos^{-1}(1) = 0° \pm 360°m \quad \text{or} \quad 0 \pm 2\pi m \text{ radians}$$

Thus $\theta_m = \cos^{-1}\left(\frac{0}{2\pi}\right) = 90°$

$\theta_m = \cos^{-1}\left(\frac{2\pi}{2\pi}\right) = \cos^{-1}(1) = 0°$

$\theta_m = \cos^{-1}\left(-\frac{2\pi}{2\pi}\right) = \cos^{-1}(-1) = 180°$

There exist two more maxima and one occurs between $53.31°$ and $66.27°$ and the other between $113.73°$ and $126.69°$. Alternate methods can be used to find them. One method will be to take derivative of $(AF)_3$ and equate it to zero. This allows you to find the extrema (minima and maxima). If the value of the second derivative at these points is positive, they are minima if it is negative they are maxima. Thus

$$\frac{d(AF)_3}{d\theta} = -\sin(2\pi\cos\theta_m)(-2\pi\sin\theta_m) = 0 \Rightarrow \sin\theta_m = 0, \sin(2\pi\cos\theta_m) = 0$$

$\sin\theta_m = 0 \Rightarrow \theta_m = \sin^{-1}(0) = 0°, 180°$. They are maxima (from above).

$\sin(2\pi\cos\theta_m) = 0 \Rightarrow 2\pi\cos\theta_m = \sin^{-1}(0) = \pm m\pi$, $m = 0, 1, 2, \cdots$

$m = 0$; $\cos\theta_m = 0 \Rightarrow \theta_m = \cos^{-1}(0) = 90°$ (Maximum)

$m = 1$; $2\pi\cos\theta_m = \pm\pi \Rightarrow \theta_m = \cos^{-1}(\pm\frac{1}{2}) = 60°, 120°$ (Maxima)

$m = 2$; $2\pi\cos\theta_m = \pm 2\pi \Rightarrow \theta_m = \cos^{-1}(\pm 1) = 0°, 180°$ (Maxima)

Therefore the maxima occur at $\theta = 0°, 60°, 90°, 120°$, and $180°$

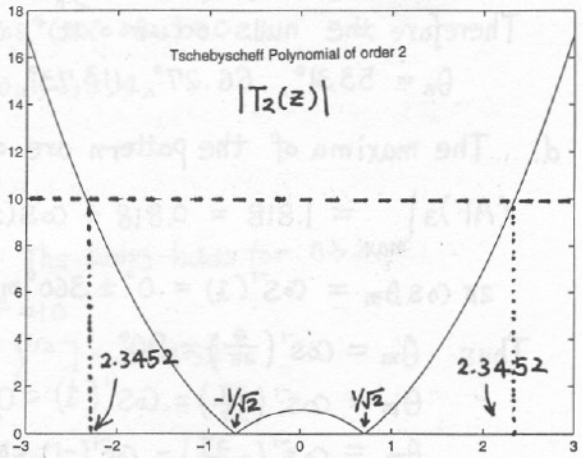
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Note: This procedure did not yield any nulls because the nulls occur at points (see graph next page) where the smooth curve was truncated and where there are zero crossings.

(Continued)

6-33 (Cont'd)

An alternate method will be to use the procedure which was used to tabulate the data of Table 6.8. Since this is a 3-element array, its array factor is equal to a Tschebyscheff polynomial of order 2, and it is shown sketched here.




The nulls occur at $z = \pm 1/\sqrt{2} = \pm 0.707$. Thus using the relation of (6-72)

$$\frac{z}{z_0} = \cos(\mu) = \cos\left(\frac{\pi}{\lambda} d \cos\theta\right)$$

For $d = \lambda$, $z = \pm 1/\sqrt{2}$, and $z_0 = 2.3452$

$$\cos(\pi\cos\theta) = \pm\frac{1}{2.3452\sqrt{2}} = \pm 0.3015$$

$$\pi\cos\theta = \cos^{-1}(\pm 0.3015) = \begin{cases} \pm 72.45^\circ = \pm 1.2645 \text{ rad} \Rightarrow \theta = \cos^{-1}\left(\pm\frac{1.2645}{\pi}\right) = 66.27^\circ \\ \qquad\qquad \text{Also } \theta = 113.73^\circ \\ \pm 107.55^\circ = \pm 1.877 \text{ rad} \Rightarrow \theta = \cos^{-1}\left(\pm\frac{1.877}{\pi}\right) = 53.31^\circ \\ \qquad\qquad \text{Also } \theta = 126.69^\circ \end{cases}$$

The maxima of the pattern occur at $z = 0$, $z = \pm z_0 = \pm 2.3452$

For $z = 0$:

$$\cos(\pi\cos\theta) = 0 \Rightarrow \pi\cos\theta = \cos^{-1}(0) = \pm\left(\frac{2m+1}{2}\right)\pi, \quad m = 0, 1, 2, \cdots$$

$m = 0$ : $\pi\cos\theta = \pm\frac{\pi}{2} \Rightarrow \theta = \cos^{-1}\left(\pm\frac{1}{2}\right) = 60^\circ, 120^\circ$

$m = 1$ : $\pi\cos\theta = \pm\frac{3\pi}{2} \Rightarrow \theta = \cos^{-1}\left(\pm\frac{3}{2}\right) \Rightarrow$ Does not exist.

The same holds for $m \geq 2$.

For $z = \pm 2.3452 = \pm z_0$:

$$\cos(\pi\cos\theta) = \frac{z}{z_0} = \pm 1 \Rightarrow \pi\cos\theta = \cos^{-1}(\pm 1) = \pm m\pi, \quad m = 0, 1, 2, \cdots$$

$m = 0$ ; $\pi\cos\theta = 0 \Rightarrow \theta = \cos^{-1}(0) = 90^\circ$

$m = 1$ : $\pi\cos\theta = \pm\pi \Rightarrow \theta = \cos^{-1}(\pm 1) = 0^\circ, 180^\circ$

$m = 2$ : $\pi\cos\theta = \pm 2\pi \Rightarrow \theta = \cos^{-1}(\pm 2) \Rightarrow$ Does not exist.

The same holds for $m \geq 3$.

Thus the same answers are obtained as with the previous method.

6-34. The procedure for this problem is identically the same as that of Problem 6.33 except that the side lobe level for this one is $-40$ dB instead of $-26$ dB.

a. $(AF)_3 = \sum_{n=1}^{2} a_n \cos[2(n-1)u] = a_1 + a_2\cos(2u) = (a_1 - a_2) + 2a_2\cos^2 u$

$R_0 = 40\,\text{dB} \Rightarrow R_0\,(\text{voltage ratio}) = 100$

$z_0 = \frac{1}{2}\left[\left(100+\sqrt{100^2-1}\right)^{1/2} + \left(100-\sqrt{100^2-1}\right)^{1/2}\right] = 7.1063$

Thus

$(a_1 - a_2) + 2a_2\cos^2 u = (a_1 - a_2) + 2a_2\left(\frac{z}{z_0}\right)^2 = 2z^2 - 1 = T_2(z)$

$2a_2\left(\frac{z}{z_0}\right)^2 = 2z^2 \Rightarrow a_2 = z_0^2 = (7.1063)^2 = 50.5$

$a_1 - a_2 = -1 \Rightarrow a_2 = z_0^2 = (7.1063)^2$

$\Rightarrow a_1 = -1 + a_2 = 49.5$

$a_1 = 49.5$, $a_2 = 50.5$ } normalized $\quad a_{1n} = a_1/a_2 = 49.5/50.5 = 0.9802$, $a_{2n} = a_2/a_2 = 50.5/50.5 = 1.0000$

b. $(AF)_3 = a_{1n} + a_{2n}\cos(2u) = 0.9802 + \cos\left(\frac{2\pi d}{\lambda}\cos\theta\right)$

or $(AF)_3 = (a_{1n} - a_{2n}) + 2a_{2n}\cos^2 u = -0.0198 + 2\cos^2\left(\frac{\pi d}{\lambda}\cos\theta\right)$

c. For $d = 3\lambda/4 \quad (0° < \theta \leq 180°)$

$(AF)_3 = 0.9802 + \cos\left[\frac{2\pi(3)}{4}\cos\theta_n\right] = 0.9802 + \cos\left(\frac{3\pi}{2}\cos\theta_n\right) = 0$

$\frac{3\pi}{2}\cos\theta_n = \cos^{-1}(-0.9802) = \begin{cases} \pm 168.58° (\pm 2.9423\,\text{rad}) \\ \pm 191.42° (\pm 3.3408\,\text{rad}) \end{cases}$

Therefore $\theta_n = \cos^{-1}\left(\pm\frac{2(2.9423)}{3\pi}\right) = 51.36°, 128.64°$

$\theta_n = \cos^{-1}\left(\pm\frac{2(3.3408)}{3\pi}\right) = 44.85°, 135.15°$

d. $D_0 = \dfrac{2R_0^2}{1 + (R_0^2 - 1)f\left(\frac{\lambda}{L+d}\right)}$, $f = 1 + 0.636\left\{\frac{2}{R_0}\cosh\left[\sqrt{(\cosh^{-1}R_0)^2 - \pi^2}\right]\right\}^2$

$f = 1 + 0.636\left\{\frac{2}{100}\cosh\left[\sqrt{(\cosh^{-1}(100))^2 - \pi^2}\right]\right\} = 1.323$

$D_0 = \dfrac{2(100)^2}{1 + (100^2-1) + 1.323\left(\frac{4}{9}\right)} = 3.4 = 5.32\,\text{dB}$

Using the computer program at the end of Chapter 2, $D_0 = 3.76 = 5.75\,\text{dB}$

e. $HPBW = \left\{\cos^{-1}\left[\cos\theta_0 - 0.443\left(\frac{\lambda}{L+d}\right)\right] - \cos^{-1}\left[\cos\theta_0 - 0.443\left(\frac{\lambda}{L+d}\right)\right]\right\} f\Big|_{\theta_0 = 90°}$

$= \left\{\cos^{-1}\left[0 - 0.443\left(\frac{4}{9}\right)\right] - \cos^{-1}\left[0 - 0.443\left(\frac{4}{9}\right)\right]\right\} 1.323 = 30.05°$

(Continued)

6-34 (Cont'd)

The excitation coefficients can also be found using (6-77b) or

$$a_n = \sum_{\xi=n}^{M+1} (-1)^{M-\xi+1} \cdot (z_0)^{2(\xi-1)} \frac{(\xi+M-2)!\,(2M)}{\epsilon_n\,(\xi-n)!\,(\xi+n-2)!\,(M-\xi+1)!} \quad \text{with } M=1$$

Thus

$$a_1 = \sum_{\xi=1}^{2} (-1)^{2-\xi} \cdot (7.1063)^{2(\xi-1)} \cdot \frac{(\xi-1)!\;2}{2(\xi-1)!\,(\xi-1)!\,(2-\xi)!} = -2 + (7.1063)^2 = 49.5$$

$$a_2 = \sum_{\xi=2}^{2} (-1)^{2-\xi} \cdot (7.1063)^{2(\xi-1)} \cdot \frac{(\xi-1)!\;2}{(\xi-2)!\,\xi!\,(2-\xi)!} = (7.1063)^2 = 50.5$$

6-35.

a. $(AF)_4 = \sum_{n=1}^{2} a_n \cos[(2n-1)u]$

$= a_1 \cos u + a_2 \cos 3u = (a_1 - 3a_2)\cos u + 4a_2 \cos^3 u$

$R_0 = 40\,dB \Rightarrow R_0 = 100$

$z_0 = \cosh\left[\frac{1}{3}\cosh^{-1}(100)\right] = 3.0095$

Therefore

$$(AF)_4 = (a_1 - 3a_2)\frac{z}{z_0} + 4a_2\left(\frac{z}{z_0}\right)^3 = -3z + 4z^3 = T_3(z)$$

$$\frac{4a_2}{(3.0095)^3} = 4 \Rightarrow a_2 = 27.257$$

$$\frac{a_1 - 3(27.257)}{(3.0095)} = -3 \Rightarrow a_1 = 72.742$$

$\Rightarrow$ normalized: $a_1 = 2.668$, $a_2 = 1$

b. $AF = 2.668\cos u + \cos 3u$, $u = \frac{\pi d}{\lambda}\cos\theta$

c. $d = \frac{3\lambda}{4}$, $u = \frac{3}{4}\pi\cos\theta$

$AF = 2.668\cos u + \cos 3u = 2.668\cos u - 3\cos u + 4\cos^3 u$

$= -0.332\cos u + 4\cos^3 u = \cos\left(\frac{3\pi}{4}\cos\theta\right)\left[-0.332 + 4\cos^2\left(\frac{3\pi}{4}\cos\theta\right)\right]$

$= \cos\left(\frac{3\pi}{4}\cos\theta\right)\left[1.668 + 2\cos\left(\frac{3\pi}{2}\cos\theta\right)\right] = 0$

$\therefore \cos\left(\frac{3\pi}{4}\cos\theta_n\right) = 0$ or $\frac{3\pi}{2}\cos\theta_n = \cos^{-1}(-0.834) = \begin{cases} 2.5571 \\ 3.7261 \end{cases}$

$\theta_n = \cos^{-1}\left(\frac{\pi}{2}\cdot\frac{4}{3\pi}\right) = 48.19^\circ$,

$\theta_n = \cos^{-1}\left[\frac{2}{3\pi}(2.5571)\right] = 57.137^\circ$

$\theta_n = \cos^{-1}\left[\frac{2}{3\pi}(3.7261)\right] = 37.7487^\circ$

Computer Result:
Directivity
$D_0 = 6.859\,dB$

Computer Program: $a_1 = 72.742$, $a_2 = 27.217$ $\Rightarrow$ Normalized $a_2 = 1$, $a_1 = 2.668$.

6-36 $(AF)_5 = \sum_{n=1}^{3} a_n \cos[2(n-1)u] = a_1\cos(0u) + a_2\cos(2u) + a_3\cos(4u)$

$= a_1 + a_2(2\cos^2 u - 1) + a_3(8\cos^4 u - 8\cos^2 u + 1)$

$= (2a_2 - 8a_3)\cos^2 u + (8\cos^4 u)a_3 + (a_1 - a_2 + a_3)$

$R_0 = 10$, $z_0 = \cosh\left(\frac{1}{4}\cosh^{-1}(10)\right) = 1.2933$, $T_4(z) = 8z^4 - 8z^2 + 1$

$\therefore (2a_2 - 8a_3)\left(\frac{z}{z_0}\right)^2 + 8a_3\left(\frac{z}{z_0}\right)^4 + (a_1 - a_2 + a_3) = -8z^2 + 8z^4 + 1$

$\frac{8z^4 \cdot a_3}{(1.2933)^4} = 8z^4, \rightarrow a_3 = 2.7976$

$\frac{z^2[2a_2 - 8(2.7976)]}{(1.2933)^2} = -8z^2 \rightarrow a_2 = 4.49992$

$a_1 - a_2 + a_3 = 1 \rightarrow a_1 = 2.7023$

a. $a_3 = 1$, $a_2 = 1.6085$, $a_1 = 0.966$

b. array factor.

$(AF) = 0.966 + 1.6085\cos(2u) + \cos(4u)$

$u = \frac{\pi d}{\lambda}\cos\theta$

c. $d = \frac{3\lambda}{4}$, $u = \frac{3\pi}{4}\cos\theta$.

$0 = 0.966 + 1.6085(2\cos^2 u - 1) + 8\cos^4 u - 8\cos^2 u + 1$

$= 8\cos^4 u - 4.783\cdot\cos^2 u + 0.3575$

$\therefore \cos\left(\frac{3\pi}{4}\cos\theta\right) = \pm 0.7143562028, \pm 0.2959226515$

$\therefore \cos(\theta_n) = \frac{4}{3\pi}\cos^{-1}(0.7143562028)$

$\theta_n = 70.79°$,

$\cos(\theta_n) = \frac{4}{3\pi}\cos^{-1}(\pm 0.2959226515)$

$\theta_n = 57.37°$, $37.423°$.

Null degree

$\theta_n = 37.423°, 57.37°, 70.79°$

Computer Result: $a_1 = 2.702$, $a_2 = 4.5$, $a_3 = 2.7$.

After Normalized: $a_1 = 1$, $a_2 = 1.609$, $a_3 = 1.0$..

6-37. $(AF)_6 = \sum_{n=1}^{3} a_n \cos[(2n-1)u] = a_1\cos u + a_2\cos 3u + a_3\cos 5u$

$R_0(dB) = 20 = 20\log_{10}(R_0) \quad \therefore R_0 = 10$

$z_0 = \cosh\left(\frac{1}{4}\cosh^{-1}(10)\right) = 1.2933.$

$(AF)_6 = a_1\cos u + a_2(4\cos^3 u - 3\cos u) + a_3(16\cos^5 u - 20\cos^3 u + 5\cos u)$

$= a_3(16)\cos^5 u + (4a_2 - 20a_3)\cos^3 u + (a_1 - 3a_2 + 5a_3)\cos u$

$= 16z^5 - 20z^3 + 5z$

$\frac{a_3(16)}{(1.2933)^5} = 16, \quad \rightarrow a_3 = 3.618$

$\frac{4a_2 - 20(3.618)}{(1.2933)^3} = -20, \quad \rightarrow a_2 = 7.275$

$\frac{a_1 - 3(7.275) + 5(3.618)}{(1.2933)} = 5 \rightarrow a_1 = 10.2015$

a. $a_1 = 2.81965, \quad a_2 = 2.011, \quad a_3 = 1$

b. $(AF)_6 = 2.81965\cdot\cos(u) + 2.011\cdot\cos(3u) + \cos(5u)$

c. Null point, $u = \frac{3\pi}{4}\cos\theta$

$(AF)_6 = 2.81965(\cos u) + 2.011(4\cos^3 u - 3\cos u) + 16\cos^5 u - 20\cos^3 u + 5\cos u$

$= 16\cos^5 u - 11.956\cos^3 u + 1.78665\cos u = 0.$

$\therefore \cos u = 0, \pm 0.7353555305, \pm 0.4544251796.$

$\cos\left(\frac{3\pi}{4}\cos\theta\right) = 0, \quad \theta_n = \cos^{-1}\left(\frac{\pi}{2}\cdot\frac{4}{3\pi}\right) = 48.19°$

$\cos\left(\frac{3\pi}{4}\cos\theta\right) = +0.735355553, \quad \theta_n = 71.57°$

$\cos\left(\frac{3\pi}{4}\cos\theta\right) = \pm 0.4544251796, \quad \theta_n = 62.19°, 29.90°$

$\therefore$ Null degree.

$\theta_n = 29.90°, 62.19°, 48.19°, 71.57°$

6-38. $R_o = 40\,dB \Rightarrow R_o$ (Voltage ratio) $= 100$

$(AF)_4 = \sum_{n=1}^{2} a_n \cos[(2n-1)u]$, $u = \frac{\pi d}{\lambda}\sin\theta\cos\phi = \frac{\pi}{4}\sin\theta\cos\phi$

$= a_1\cos u + a_2\cos 3u = (a_1 - 3a_2)\cos u + 4a_2\cos^3 u$

$z_0 = \cosh\left[\frac{1}{3}\cosh^{-1}(100)\right] = 3.0095$

Therefore

$(AF)_4 = (a_1 - 3a_2)\frac{z}{z_0} + 4a_2\left(\frac{z}{z_0}\right)^3 = -3z + 4z^3 \Rightarrow \frac{a_1 - 3a_2}{z_0} = -3,\ \frac{4a_2}{z_0^3} = 4$

$a_2 = z_0^3 = (3.0095)^3 = 27.2573$, $a_1 = -3z_0 + 3a_2 = 72.7434$

In normalized form $a_{2n} = a_2/a_2 = 1.00$, $a_{1n} = a_1/a_2 = 2.6688$

$(AF)_4 = 2.6688\cos(\frac{\pi}{4}\sin\theta\cos\phi) + \cos(\frac{3\pi}{4}\sin\theta\cos\phi)$

$E_{total} = j\eta\frac{I_o e^{-jkr}}{2\pi r}\cdot\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\left[2.6688\cos(\frac{\pi}{4}\sin\theta\cos\phi) + \cos(\frac{3\pi}{4}\sin\theta\cos\phi)\right]$

a. $D_o = \left[\frac{2R_o^2}{1+(R_o^2-1)f(\frac{\lambda}{L+d})}\right] D(l=\frac{\lambda}{2})$, where $D(l=\frac{\lambda}{2}) = 1.65$

$f = 1. + 0.636\left\{\frac{2}{R_o}\cosh\left[\sqrt{(\cosh^{-1}(R_o))^2 - \pi^2}\right]\right\}^2$

$= 1 + 0.636\left\{\frac{2}{100}\cosh\left[\sqrt{(5.2983)^2 - \pi^2}\right]\right\}^2 = 1.323$

$D_o = \frac{2(100)^2(1.65)}{1+(100^2-1)1.323\left(\frac{\lambda}{\frac{3\lambda}{4}+\frac{\lambda}{4}}\right)} = 2.49 = 3.96\,dB$, $D_o$ (Computer) $= 3.2 = 5.05\,dB$

b. $\theta = \pi/2$ (x-y plane)

$E_\theta(total)\Big|_{3\text{-}dB} = (1)[2.6688\cos(\frac{\pi}{4}\cos\phi) + \cos(\frac{3\pi}{4}\cos\phi)] = \frac{3.6688}{\sqrt{2}} \Rightarrow \phi = 54.45^\circ$

$HPBW = 2(90 - 54.45) = 71.1^\circ$

$\phi = \pi/2$ (y-z plane)

$E_\theta(total)\Big|_{3\text{-}dB} = 3.6688\left[\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right] = \frac{3.6688}{\sqrt{2}} \Rightarrow \theta = 50.8^\circ$

$HPBW = 2(90 - 50.8) = 78.4^\circ$ which is the same as that of a $\lambda/2$ dipole.

6-39. The answers for this are identically the same as those of Problem 6.38 except that the $\sin\theta\cos\phi$ factor is replaced by the $\sin\theta\sin\phi$ factor. Also the HPBW in the x-Zplane of this problem is the same as that of the y-z plane of Prob 6-38.

6-40. $(AF)_5 = \sum_{n=1}^{3} a_n \cos[2(n-1)u] = a_1 + a_2\cos(2u) + a_3\cos(4u)$

$= a_1 + a_2(2\cos^2 u - 1) + a_3(8\cos^4 u - 8\cos^2 u + 1)$

a. $R_0 = 40\text{dB} \Rightarrow R_0\,(\text{Voltage ratio}) = 100$

$$Z_0 = \frac{1}{2}\left\{\left[100 + \sqrt{100^2-1}\,\right]^{1/4} + \left[100 - \sqrt{100^2-1}\,\right]^{1/4}\right\} = 2.013248$$

letting $\cos u = z/z_0$

$(AF)_5 = (a_1 - a_2 + a_3) + (2a_2 - 8a_3)\cos^2 u + 8a_3\cos^4 u$

$= (a_1 - a_2 + a_3) + (2a_2 - 8a_3)\left(\frac{z}{z_0}\right)^2 + 8a_3\left(\frac{z}{z_0}\right)^4 = 1 - 8z^2 + 8z^4$

Equating alike terms yields $a_3 = 16.429$, $a_2 = 49.503$, $a_1 = 34.074$

or in normalized form $a_{3n} = a_3/a_3 = 1.0$, $a_{2n} = a_2/a_3 = 3.013$, $a_{1n} = a_1/a_3 = 2.074$

b. $(AF)_5 = 2.074 + 3.013\cos\left(\frac{\pi}{2}\sin\theta\cos\phi\right) + \cos(\pi\sin\theta\cos\phi)$, $u = \frac{\pi d}{\lambda}\sin\theta\cos\phi\Big|_{d=\lambda/4}$

c. $f = 1 + 0.636\left\{\frac{2}{100}\cosh\left[\sqrt{(\cosh^{-1}100)^2 - \pi^2}\right]\right\}^2 = 1 + 0.636\left[\frac{2}{100}(35.64)\right]^2 = 1.323$

$$D_0 = \frac{2R_0^2}{1 + (R_0^2 - 1)f\left(\frac{\lambda}{L+d}\right)} = \frac{2(100)^2}{1 + (100^2-1)\,1.323\,\frac{1}{1.25}} = 1.889 = 2.76\text{dB}$$

$D_0$ (Computer Program) $= 1.998 = 3.01\text{dB}$
at the end of Chapter 2

d. $HPBW = f\left\{\cos^{-1}\left(\cos\theta_0 - 0.443\frac{\lambda}{L+d}\right) - \cos^{-1}\left(\cos\theta_0 + 0.443\frac{\lambda}{L+d}\right)\right\}_{\theta_0 = 90^\circ}$

$= 1.323\left\{\cos^{-1}\left[-0.443\left(\frac{1}{1.25}\right)\right] - \cos^{-1}\left[0.443\left(\frac{1}{1.25}\right)\right]\right\} = 1.323(41.513^\circ)$

$HPBW = 54.9^\circ$

6-41. a. $N = 2M+1 = 9 \Rightarrow M = 4$, $R_o = 30\,dB \Rightarrow R_o(\text{voltage Ratio}) = 10^{1.5} = 31.662$

$$z_0 = \frac{1}{2}\left\{\left[31.662 + \sqrt{(31.662)^2 - 1}\right]^{1/8} + \left[31.662 - \sqrt{(31.662)^2 - 1}\right]^{1/8}\right\}$$

$$z_0 = 1.679244 + 0.595506 = 1.137375$$

b. $(AF)_9 = \sum_{n=1}^{5} a_n \cos[2(n-1)u] = a_1 + a_2\cos(2u) + a_3\cos(4u) + a_4\cos(6u) + a_5\cos(8u)$

$$= a_1 + a_2(2\cos^2 u - 1) + a_3(8\cos^4 u - 8\cos^2 u + 1) + a_4(32\cos^6 u - 48\cos^4 u + 18\cos^2 u - 1) + a_5(128\cos^8 u - 256\cos^6 u + 160\cos^4 u - 32\cos^2 u + 1)$$

$$(AF)_9 = (a_1 - a_2 + a_3 - a_4 + a_5) + (2a_2 - 8a_3 + 18a_4 - 32a_5)\cos^2 u + (8a_3 - 48a_4 + 160a_5)\cos^4 u + (32a_4 - 256a_5)\cos^6 u + (128a_5)\cos^8 u$$

letting $z/z_0 = \cos u$ and equating $(AF)_9 = T_8(z) = 1 - 32z^2 + 160z^4 - 256z^6 + 128z^8$

leads to $a_5 = z_0^8 = (1.13737)^8 = 2.8004$

$$-256 = \frac{32a_4 - 256a_5}{z_0^6} \Rightarrow a_4 = (-256 z_0^6 + 256a_5)/32 = 5.085$$

$$160 = \frac{8a_3 - 48a_4 + 160a_5}{z_0^4} \Rightarrow a_3 = (160 z_0^4 + 48a_4 - 160a_5)/8 = 7.970$$

$$-32 = \frac{2a_2 - 8a_3 + 18a_4 - 32a_5}{z_0^2} \Rightarrow a_2 = (-32z_0^2 + 8a_3 - 18a_4 + 32a_5)/2 = 10.226$$

$$1 = a_1 - a_2 + a_3 - a_4 + a_5 \Rightarrow a_1 = 1 + a_2 - a_3 + a_4 - a_5 = 5.540$$

Thus $a_1 = 5.540 \Rightarrow 2a_1 = 11.080$
$a_2 = 10.226$
$a_3 = 7.970$
$a_4 = 5.085$
$a_5 = 2.8004$

In Normalized Form $\Rightarrow$ $a_{1n} = a_1/a_5 = 1.978$, $2a_{1n} = 3.956$
$a_{2n} = a_2/a_5 = 3.652$
$a_{3n} = a_3/a_5 = 2.846$
$a_{4n} = a_4/a_5 = 1.815$
$a_{5n} = a_5/a_5 = 1.0$

c. $f = 1 + 0.636\left\{\frac{2}{31.662}\cosh\left[\sqrt{(\cosh^{-1}31.662)^2 - \pi^2}\right]\right\}^2 = 1 + 0.636\left[\frac{2}{31.662}(7.5373)\right]^2 = 1.144$

$$D_o = 2R_o^2 / \left\{1 + (R_o^2 - 1)f\left(\frac{\lambda}{L+d}\right)\right\} = 2(31.662)^2 / \left\{1 + (31.662^2 - 1)\,1.144\left(\frac{1}{4.5}\right)\right\}$$

$$= 7.844 = 8.945\,dB$$

d. $HPBW = f\left\{\cos^{-1}\left(\cos\theta_o - 0.443\frac{\lambda}{L+d}\right) - \cos^{-1}\left(\cos\theta_o + 0.443\frac{\lambda}{L+d}\right)\right\}_{\theta_o = 90^\circ}$

$$= 1.144\left\{\cos^{-1}\left(-0.443\frac{1}{4.5}\right) - \cos^{-1}\left(0.443\frac{1}{4.5}\right)\right\} = 1.144(95.65 - 84.35)$$

$$HPBW = 1.144\,(11.30) = 12.93^\circ$$

6-42 $2N+1 = 3,\ N = 1$

a. $(AF)_{2N+1} = \sum_{n=1}^{N+1} a_n \cos[2(n-1)u] = \sum_{n=1}^{2} a_n \cos[2(n-1)u] = a_1 + a_2\cos(2u)$

$= a_1 + a_2\cos(2u) = a_1 + a_2[2\cos^2 u - 1] = (a_1 - a_2) + 2a_2\cos^2 u$

$(AF)_{2N+1} = (a_1 - a_2) + 2a_2\cos^2 u$

$R_0(dB) = 26 = 20\log_{10}[R_0(\text{dimensionless})]$, $1.3 = \log_{10} R_0$

$R_0 = 10^{1.3} = 20$,

$20 = T_2(z_0) = 2z_0^2 - 1$, $2z_0^2 = 21$, $z_0^2 = 10.5$, $z_0 = 3.24$

$\cos u = z/z_0$

$(AF)_{2N+1} = (a_1 - a_2) + 2a_2\cos^2 u = T_2(z) = 2z^2 - 1$

$(a_1 - a_2) + 2a_2(z/z_0)^2 = 2z^2 - 1$, $\frac{2a_2}{z_0^2} = 2 \Rightarrow a_2 = (z_0^2) = (3.24)^2 = 10.5$

$(a_1 - a_2) = -1$, $a_1 = 10.5 - 1 = 9.5$, $a_1 = 9.5 \Rightarrow 2a_1 = 19$

$a_{1n} = 0.905 \Rightarrow 2a_{1n} = 1.81$, $a_{2n} = 1.0$, $a_{2n} = 1.0$

| $a_2$ | $2a_1$ | $a_2$ |
|---|---|---|
| 10.5 | 19 | 10.5 |
| 1.0 | 1.81 | 1.0 |
| 0.553 | 1 | 0.553 |

b. $(AF)_{2N+1} = (AF)_3 = \sum_{n=1}^{2} a_n \cos[2(n-1)u] = a_1 + a_2\cos(2u)$

$(AF)_3 = 0.905 + \cos[2u] = 0.905 + \cos\left[\frac{2\pi}{\lambda} d\cos\theta\right]$

$= 0.905 + \cos\left[\frac{2\pi}{\lambda} d\cos\theta\right]$

$= 0.5 + 0.553\cos\left(\frac{2\pi}{\lambda} d\cos\theta\right)$

(Continued)

6-42 (Cont'd)

c. $(AF)_3 = 0.905 + \cos\left(\frac{2\pi}{\lambda} d\cos\theta_n\right)\Big|_{d=\frac{\lambda}{2}} = 0$

$\cos\left(\frac{2\pi}{\lambda}\frac{\lambda}{2}\cos\theta_n\right) = \cos(\pi\cos\theta_n) = -0.905$

$\pi\cos\theta_n = \cos^{-1}(-0.905) = \begin{cases} \pm 154.82^\circ \text{ or } \pm 2.702 \text{ rad} \\ \pm 205.176^\circ \text{ or } \pm 3.581 \text{ rad} \end{cases}$

$\theta_1 = \cos^{-1}\left(\pm\frac{2.702}{\pi}\right) = \cos^{-1}(\pm 0.86007) = 30.675^\circ, 149.316^\circ$

$\theta_1 = \cos^{-1}\left(\pm\frac{3.581}{\pi}\right) = \cos^{-1}(1.13986) = \text{does not exist.}$

$\theta = 30.675^\circ, 149.316^\circ$

d. $(AF)_3 = 0.905 + \cos\left(\frac{2\pi}{\lambda} d\cos\theta_m\right)_{d=\lambda/2} = 0.905 + \cos(\pi\cos\theta_m) = 1.905$

$\cos(\pi\cos\theta_m) = 1, \quad \pi\cos\theta_m = \cos^{-1}(1) = 2m\pi, \quad m = 0, \pm 1, \pm 2, \pm 3, \cdots$

$\cos\theta_m = 2m \Rightarrow \theta_m = \cos^{-1}(2m)$

$m=0: \quad \theta_m = \cos^{-1}(0) = 90^\circ \Big\} \quad \theta = 90^\circ.$

e. $R_0 = 20, \quad f = 1 + 0.636\left\{\frac{2}{R_0}\cosh\left[\sqrt{(\cosh^{-1}R_0)^2 - \pi^2}\right]\right\}^2$

$\cosh^{-1}(R_0) = \ln\left[R_0 \pm (R_0^2 - 1)^{1/2}\right] = \ln\left[20 \pm (400-1)^{1/2}\right] = 3.688\,25$

$f = 1 + 0.636\left\{\frac{2}{20}\cosh\left(\sqrt{3.7336}\right)\right\}^2 = 1 + 0.636\left[\frac{1}{10}\cosh(1.93225)\right]^2$

$f = 1 + 0.079 = 1.079$

$\theta_n = 2\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{1.391\lambda}{\pi N d}\right)\right] = 2\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{1.391\lambda}{\pi \cdot 3 \cdot \frac{\lambda}{2}}\right)\right] = 2\left[\frac{\pi}{2} - \cos^{-1}(0.295)\right]$

$\theta_n = 2\left[\frac{\pi}{2} - 1.271\right] = 0.5996 \text{ rad} = 34.3545^\circ$

$\theta_n' = \theta_n f = 34.3545^\circ(1.079) = 37.068^\circ$

f. $D_0 = \frac{2R_0^2}{1 + (R_0^2 - 1)f\left(\frac{\lambda}{L+d}\right)} = \frac{2(20)^2}{1 + (20^2-1)\,1.079\,\frac{\lambda}{1.5\lambda}} = 2.7776$

$D_0 = 2.7776 = 4.436 \text{ dB}$

6-43. a. $D_o = 2N\left(\frac{d}{\lambda}\right)$

$D_o = 33\text{dB} = 10\log_{10} D_o$ (dimensionless)

$3.3 = \log_{10} D_o$ (dimensionless)

$D_o$ (dimensionless) $= 10^{3.3} = 1{,}995.26$

$1995.26 = 2N\left(\frac{d}{\lambda}\right) = 2N\left(\frac{\lambda}{16\lambda}\right) = \frac{N}{8} \Rightarrow N = 15{,}962.1 \simeq 15{,}962.$

$N = 15{,}962$

b. $L = (N-1)d = (15{,}962-1)\frac{\lambda}{16} = \frac{(15{,}961\cdot\lambda)}{16} = 997.56\,\lambda$

$L = 997.56\,\lambda$

c. $\text{HP}_h \simeq 2\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{1.391\lambda}{\pi N d}\right)\right] = 2\left[90° - \cos^{-1}\left(\frac{1.391(16)\lambda}{\pi(15{,}962)\lambda}\right)\right]$

$\text{HP}_h \simeq 2[90° - 89.9745°] = 0.05086° = 0.05086°$

d. $-13.46 \simeq -13.5\,\text{dB}$

6-44. a. $30\text{dB} = 20\log_{10} R_o(\text{VR})$, $\quad \cosh^{-1}(31.623) = \ln\left[31.613 + \sqrt{(31.613)^2 - 1}\right]$

$= 4.147$

$R_o(\text{VR}) = 10^{1.5} = 31.623$,

$f = 1 + 0.636\left\{\frac{2}{31.623}\cosh\left[\sqrt{(\cosh^{-1}(31.623))^2 - \pi^2}\right]\right\}^2$

$= 1 + 0.636\left\{\frac{2}{31.623}\cosh[2.707]\right\}^2 = 1 + 0.636\left\{\frac{2}{31.623}(7.5257)\right\}^2$

$= 1 + 0.636(0.226) = 1 + 0.144 = 1.144$

$f = 1.144$

$\text{HP}_h = 1.144\,(0.0586) = 0.067°$

$\text{HP}_d \simeq \frac{101.5}{D_o} = \frac{101.5}{1{,}995.26} = 0.05087 \simeq 0.0509$

$\text{HP}_h = 1.144\,(0.0509) = 0.0582$

b. $D_o = \frac{2R_o^2}{1 + (R_o^2 - 1)f\frac{\lambda}{L+d}} = \frac{2(31.623)^2}{1 + [(31.623)^2 - 1]1.144\frac{\lambda}{(997.56 + \frac{1}{16})\lambda}}$

$= \frac{2(31.623)^2}{1 + 1.146} = 931.98 = 29.694\,\text{dB}$

$D_o = 931.98 = 29.694\,\text{dB}$

6-45. a. Number of Elements = Number of complete minor lober + 2

$= 4 + 2 = 6$

b. Excitation coefficients $a_n$. $2M = 6 \Rightarrow M = 3$, $P = 6 - 1 = 5$

$-40dB$ sidelobe level : $\Rightarrow R_0(dB) = 40 = 20\log_{10} R_0(VR) \Rightarrow R_{0VR} = 10^2 = 100$

$$Z_0 = \frac{1}{2}\left[\left(R_{0VR} + \sqrt{R_{0VR}^2 - 1}\right)^{1/P} + \left(R_{0VR} - \sqrt{R_{0VR}^2 - 1}\right)^{1/P}\right]$$

$$= \frac{1}{2}\left[\left(100 + \sqrt{100^2 - 1}\right)^{1/5} + \left(100 - \sqrt{100^2 - 1}\right)^{1/5}\right]$$

$$= \cosh\left[\frac{1}{P}\cosh^{-1}(R_{0VR})\right] = \cosh\left[\frac{1}{5}\cosh^{-1}(100)\right] = 1.6159$$

$$a_n = \sum_{\xi=n}^{M} (-1)^{M-\xi} (Z_0)^{2\xi-1} \frac{(\xi+M-2)!\,(2M-1)}{(\xi-n)!\,(\xi+n-1)!\,(M-\xi)!}$$

$$n=1;\quad a_1 = \sum_{\xi=1}^{3} (-1)^{3-\xi} (Z_0)^{2\xi-1} \cdot \frac{(\xi+3-2)!\,(2(3)-1)}{(\xi-1)!\,(\xi+1-1)!\,(3-\xi)!}$$

$$= (-1)^2 Z_0 \frac{2!\,5}{0!\,1!\,2!} + (-1)^1 \frac{Z_0^3\, 3!\,5}{1!\,2!\,1!} + (-1)^0 Z_0^5 \frac{4!\,5}{2!\,3!\,0!}$$

$$= 8.0795 - 63.79 + 110.172 = 54.96$$

$$n=2;\quad a_2 = \sum_{\xi=2}^{3} (-1)^{3-\xi} (Z_0)^{2\xi-1} \frac{(\xi+3-2)!\,(2(3)-1)}{(\xi-2)!\,(\xi+2-1)!\,(3-\xi)!}$$

$$= (-1)^1 Z_0^3 \frac{3!\,(5)}{0!\,3!\,0!} + (-1)^0 Z_0^5 \frac{4!\,5}{1!\,4!\,0!} = -21.097 + 55.086 = 33.9891$$

$$n=3;\quad a_3 = \sum_{\xi=3}^{3} (-1)^{3-\xi} (Z_0)^{2\xi-1} \frac{(\xi+3-2)!\,[2(3)-1]}{(\xi-3)!\,(\xi+3-1)!\,(3-\xi)!} = (-1)^0 Z_0^5 \frac{4!\,5}{0!\,5!\,0!} = 11.0172$$

$a_1 = 54.96$
$a_2 = 33.9891$
$a_3 = 11.0172$

$\Longrightarrow$ Normalized

$a_1 = 4.9886$
$a_2 = 3.0851$
$a_3 = 1$

c. $$d = \frac{\lambda}{\pi}\cos^{-1}\left(-\frac{1}{Z_0}\right) = \frac{\lambda}{\pi}\cos^{-1}\left(-\frac{1}{1.6159}\right) = 0.7124\,\lambda$$

d. $$(AF)_n = a_1\cos(\mu) + a_2\cos(3\mu) + a_3\cos(5\mu),$$

$$\mu = \frac{\pi d}{\lambda}\cos\theta = \pi\left(\frac{0.7\cdot\lambda}{\lambda}\right)\cos\theta = 0.7\cdot\pi\cdot\cos\theta$$

$$\mu = 2.1991\cos\theta$$

6-45 (Cont'd) $(AF)_6 = 4.988\cos(u) + 3.085\cos(3u) + \cos(5u)$

where $u = \frac{\pi d}{\lambda}\cos(\theta)$

Spacing d $d_{max} \leq \frac{\lambda}{\pi}\cos^{-1}(-\frac{1}{Z_0})$

for the condition which all the minor lobes to be of the same level. The $d_{max} = 0.71239\lambda$ is obtained by using the above equation. But one problem occurs when we want to use that spacing. The radiation pattern of the array factor shows that another half minor lobes existed in the zero degree when $d = 0.71239\lambda$. In order to meet the specification that exactly four minor lobes are expected in the region $(0° \leq \theta \leq 90°)$, the maximum spacing need to be modified under this condition. After some trying, the maximum eligible spacing for the specification is $0.7\lambda$, Tus, let $d = 0.7\lambda$ in this design.
The radiation pattern is drawn in range between 0dB and -70 dB in Fig. P6-49.

Array Factor Power Pattern of a 6-element Broadside
Doplh-Tschebycheff Array Antenna

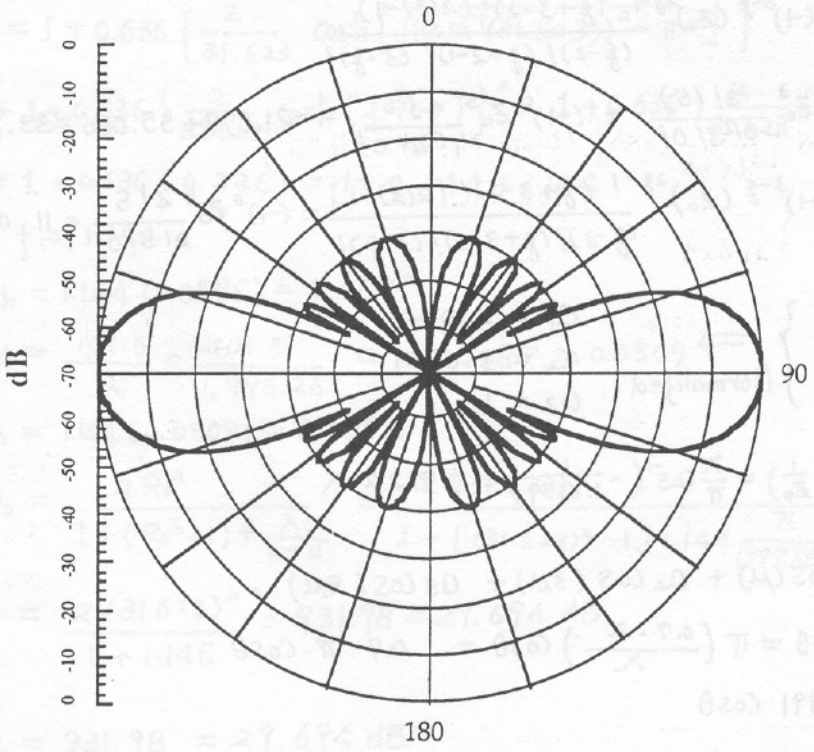



Fig. P6-45 FOUR SIDE LOBES ALL AT -40 dB LEVEL

6-46

a. Tschebyscheff

b. $30 = 20\log_{10}(R_{OVR}) \Rightarrow R_{OVR} = 10^{3/2} = 31.623$

$$z_0 = \frac{1}{2}\left[\left(R_0 + \sqrt{R_0^2-1}\right)^{1/p} + \left(R_0 - \sqrt{R_0^2-1}\right)^{1/p}\right]$$

$$= \frac{1}{2}\left[\left(31.623 + \sqrt{(31.623)^2-1}\right)^{1/2} + \left(31.623 - \sqrt{(31.623)^2-1}\right)^{1/2}\right]$$

$$= \frac{1}{2}\left[(31.623+31.607)^{1/2} + (31.623-31.607)^{1/2}\right]$$

$$z_0 = \frac{1}{2}[7.952 + 0.1265] = 4.039$$

$2M+1 = 3 \Rightarrow M = 1, \quad M+1 = 2$

$$a_1 = (-1)^{1-1+1}(4.039)^{2(1-1)}\frac{(1+2-2)!\,2}{2(1-1)!(1+1-2)!(1-1+1)!} + (-1)^{1-2+1}(4.039)^{2(2-1)}\frac{(2+1-2)!\,2}{2(2-1)!(2+1-2)!(1-2+1)!}$$

$$= (-1)(1)\frac{1\cdot 2}{2(1)(1)} + (1)(4.039)^2\frac{1\cdot 2}{2(1)(1)(1)}$$

$$a_1 = -1 + 16.314 = 15.314$$

$$a_2 = (-1)^{1-2+1}(4.039)^{2(2-1)}\frac{(2+1-2)!\,2!}{2(2-2)!(2+2-2)!(1-2+1)!}$$

$$a_2 = (1)(4.039)^2\frac{(1)(2)}{2(1)(2)(1)} = 16.314/2 = 8.157$$

c. $$f = 1 + 0.636\left\{\frac{2}{31.623}\cosh\left[(\cosh^{-1}31.623)^2 - \pi^2\right]\right\}^2$$

$$= 1 + 0.636\left\{\frac{2}{31.623}\cosh\left[\sqrt{(4.1468)^2-\pi^2}\right]\right\}^2$$

$$f = 1 + 0.636\left\{\frac{2}{31.623}\cosh(2.707)\right\}^2 = 1 + 0.636\left\{\frac{2}{31.623}7.525\right\}^2$$

$f = 1.144$

$$\text{HPBW} = \Theta h = \cos^{-1}\left[\cos\theta_0 - 0.443\frac{\lambda}{L+d}\right]_{\theta_0=90^\circ} - \cos^{-1}\left[\cos\theta_0 + 0.443\frac{\lambda}{L+d}\right]_{\theta_0=90^\circ}$$

$$= \cos^{-1}\left[-0.443\frac{\lambda}{3\lambda/4}\right] - \cos^{-1}\left[0.443\frac{\lambda}{3\lambda/4}\right] = 126.20^\circ - 53.8^\circ$$

$\Theta h = 72.4^\circ$

$\Theta h\,(Tsc) = 72.4(1.144) = 82.8256$

d. $$D_0 = \frac{2(1,000)}{1 + (999)\,1.144\cdot\frac{4}{3}} = 1.312 = 1.1793\,dB$$

6-47.

$$a_n = 1 + \cos\left(\frac{\pi x_n}{L}\right)$$

$$2a_1 = 1 + \cos\left(\frac{\pi x_1}{L}\right) = 1 + \cos(0) = 2 \Rightarrow a_1 = 1$$

$$a_2 = 1 + \cos\left(\frac{\pi x_2}{L}\right)\Big|_{x_2 = L/4} = 1 + \cos\left(\frac{\pi L}{4L}\right) = 1 + \cos\left(\frac{\pi}{4}\right) = 1.707$$

$$a_3 = 1 + \cos\left(\frac{\pi x_3}{L}\right)_{x_3 = L/2} = 1 + \cos\left(\frac{\pi}{2}\right) = 1.$$

$\therefore$ $a_1 = 1$

$a_2 = 1.707$

$a_3 = 1$

6-48 The length of the array is obtained using an iterative procedure of (6-22) or its graphical solution of Figure 6.11.

$$\theta_h = \cos^{-1}\left(\cos\theta_0 - \frac{2.782}{Nkd}\right) - \cos^{-1}\left(\cos\theta_0 + \frac{2.782}{Nkd}\right)$$

a. $\theta_0 = 30^\circ$, using iterative procedure

$$\frac{2^\circ}{180^\circ} \times \pi = \cos^{-1}\left(\cos\frac{\pi}{6} - \frac{2.782}{x}\right) - \cos^{-1}\left(\cos\frac{\pi}{6} + \frac{2.782}{x}\right)$$

$Nkd = x = 318.95$, $Nkd = 318.95$, $N = (L+d)/d$

$\therefore (L+d)\frac{2\pi}{\lambda} = 318.95$, $L + d = 50.76\lambda$ $\therefore L = (50.76\lambda - d)$

From Fig 6.11, $L + d = 50\lambda$ $\therefore L = (50\lambda - d)$

b. $\theta_0 = 45^\circ$, using iterative procedure

$Nkd = x = 225.46$, $L + d = 35.88\lambda$ $\therefore L = (35.88\lambda - d)$

From Fig 6.11, $L + d = 35\lambda$ $\therefore L = (35\lambda - d)$

c. $\theta_0 = 60^\circ$, using iterative procedure

$Nkd = x = 184.07$, $L + d = 29.2956\lambda$, $L = 29.2956\lambda - d$

From Fig. 6.11, $L + d = 30\lambda$, $\therefore L = (30\lambda - d)$.

6-49. $\theta_0 = 60°$, $\phi_0 = 90°$, $dx = dy = \lambda$, $0 \le \theta \le 180°$, $0° \le \phi \le 360°$

We need to find $\phi$ and $\theta$ which angle satisfy both (6-49a) and (6-49b) ie

$$\sin\theta\cos\phi - \sin\theta_0 \cdot \cos\phi_0 = \pm \frac{m\lambda}{dx} = \pm m \quad (\leftarrow dx = \lambda)$$

$$\sin\theta \cdot \sin\phi - \sin\theta_0 \cdot \sin\phi_0 = \pm \frac{n\lambda}{dy} = \pm n \quad (\leftarrow dy = \lambda)$$

$$\sin\theta\cos\phi = \pm m \quad ; \quad m = -1, 0, 1$$

$$\sin\theta \cdot \sin\phi - \frac{\sqrt{3}}{2} = \pm n \quad ; \quad n = 0, 1$$

In case of

n=0

- m=-1 ; $\sin\theta\cos\phi = -1$ : → $\theta = 90°$ and $\phi = 180°$
  $\sin\theta\sin\phi = \frac{\sqrt{3}}{2}$ : $\sin 90° \cdot \sin 180° \neq \frac{\sqrt{3}}{2}$
- m=0 : $\sin\theta\cos\phi = 0$ : ① → $\theta = 0°$ or $\theta = 180°$ or $\phi = 90°$ or $\phi = 270°$
  $\sin\theta\sin\phi = \frac{\sqrt{3}}{2}$ : ② ① ≠ ② ① ≠ ② $\theta = 60°, 120°$ ① = ② $\theta = 240°, 300°$ ① = ②
- m=+1 : $\sin\theta\cos\phi = 1$ : ① → ($\theta = 90°$ and $\phi = 0°$) or ($\theta = 90°$ and $\phi = 360°$)
  $\sin\theta\sin\phi = \frac{\sqrt{3}}{2}$ : ② ① ≠ ② ① ≠ ②

In case of

n=-1

- m=-1 ( $\sin\theta\cos\phi = -1$ ; ① ($\theta = 90°$ and $\phi = 180°$)
  $\sin\theta\sin\phi = \frac{\sqrt{3}}{2} - 1$ ; ② ① ≠ ②
- m=0 ( $\sin\theta\cos\phi = 0$ : ① $\theta = 0°$ or $\theta = 180°$ or $\phi = 90°$ or $\phi = 270°$
  $\sin\theta\sin\phi = \frac{\sqrt{3}}{2} - 1$ : ②, ① ≠ ② , ① ≠ ② $\theta = -7.699°, -172.301°$
  ① = ② $\theta = 7.699°, 172.301°$
  ① = ②
- m=+1 ( $\sin\theta\cos\phi = +1$ : ① ($\theta = 90°$ and $\phi = 0°$) or ($\theta = 90°$ and $\phi = 360°$)
  $\sin\theta\sin\phi = \frac{\sqrt{3}}{2} - 1$ : ② ① ≠ ② ① ≠ ②

Again, we found the angle of grating lobe

n = 0, m = 0 ; At, $\phi = 90°$ < $\theta = 60°$ → Major lobe; $\theta = 120°$ → Grating lobe

$\phi = 270°$ < $\theta = 240°$, $\theta = 300°$ } ⇒ No exist in physical.

(Continued)

6-49 (Cont'd)

$n=-1, m=0$ $\begin{cases} \phi = 90^\circ \\ \theta = -7.699^\circ \\ \theta = -172.301 \end{cases}$ } No exist in physical meaning

At $\phi = 270^\circ$, $\theta = 7.699^\circ$ (↑ Grating lobe), $\theta = 172.301^\circ$ (↑ Grating lobe)

Therefore, three grating lobes exist.

$n=0, m=0$, $(\theta=120^\circ, \phi=90^\circ)$ Ⓐ

$n=-1, m=0$, $(\theta=7.699^\circ, \phi=270^\circ)$ Ⓑ $(\theta=172.301^\circ, \phi=270^\circ)$ Ⓒ

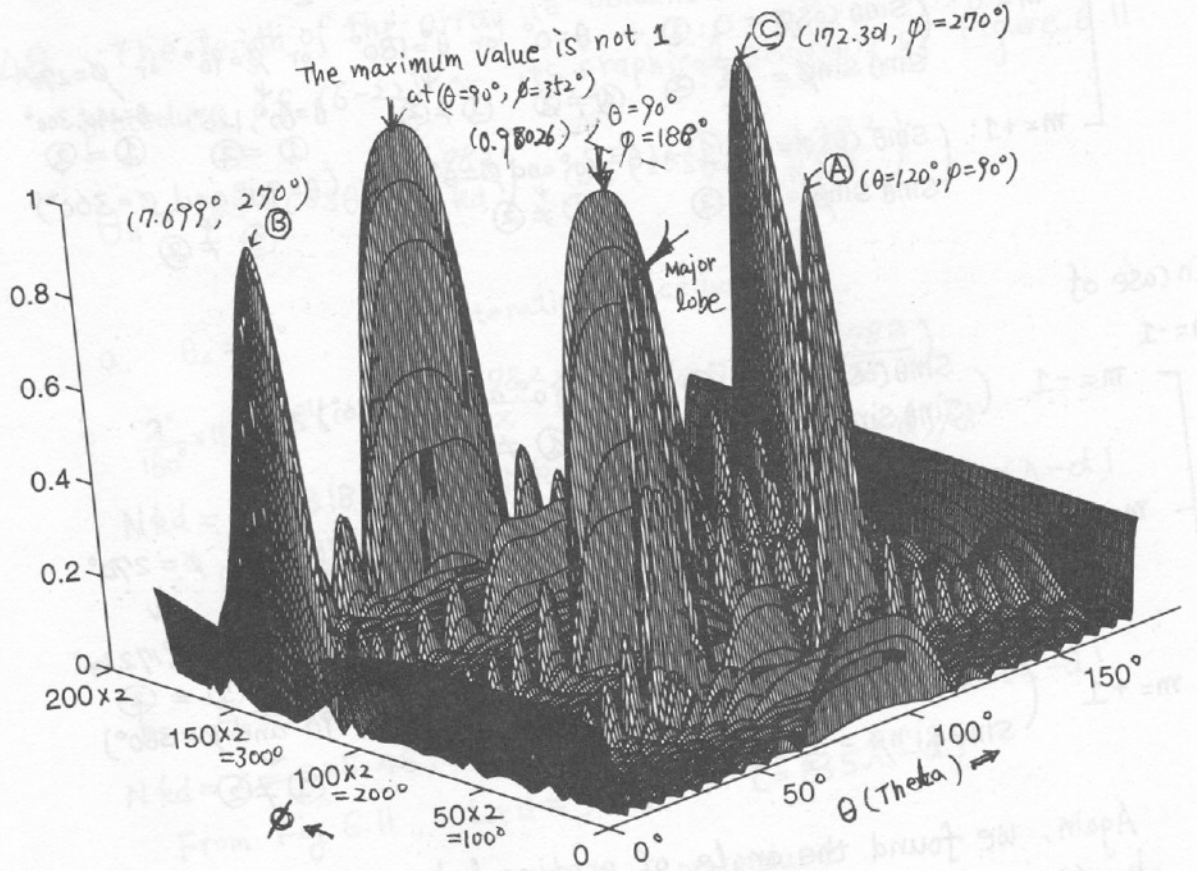



Figure P6-49.

6-50. $d_x = d_y = \lambda/8$, $M=10$, $N=8$, $\theta_0 = 10°$, $\phi_0 = 90°$

a. $\beta_x = -kd_x \sin\theta_0 \cos\phi_0 = -\frac{2\pi}{\lambda}\frac{\lambda}{8}\sin(10°)\cos(90°) = 0$

$\beta_y = -kd_y \sin\theta_0 \sin\phi_0 = -\frac{2\pi}{\lambda}\frac{\lambda}{8}\sin(10°)\sin(90°) = -0.1364 \text{ rad} = -7.81°$

b. $D_0 = \pi \cos\theta_0 \, D_x D_y$

$D_x = 2N\left(\frac{d_x}{\lambda}\right) = 2(10)\left(\frac{1}{8}\right) = 2.5 = 3.98 \text{ dB}$

$D_y = 2N\left(\frac{d_y}{\lambda}\right) = 2(8)\left(\frac{1}{8}\right) = 2.0 = 3.01 \text{ dB}$

$D = \pi\cos(10°)(2.5)(2) = 15.47 = 11.89 \text{ dB}$

c. $\Theta_{x0} = \cos^{-1}\left[\cos\theta_0 - 0.443\frac{\lambda}{L+d_x}\right]_{\theta_0=90°} - \cos^{-1}\left[\cos\theta_0 + 0.443\frac{\lambda}{L+d_x}\right]_{\theta_0=90°}$

$= \cos^{-1}\left(-0.443\frac{1}{1.25}\right) - \cos^{-1}\left(0.443\frac{1}{1.25}\right) = 110.76° - 69.24° = 41.52°$

$= 0.7245 \text{ rad}$

Also from Table 6.2

$\Theta_{x0} = 2\left[90° - \cos^{-1}\left(\frac{1.391\lambda}{\pi M d_x}\right)\right] = 2\left[90° - \cos^{-1}\left(\frac{1.391\times 8}{10\pi}\right)\right] = 41.49° = 0.724 \text{ rad}$

$\Theta_{y0} = \cos^{-1}\left(\cos\theta_0 - 0.443\frac{\lambda}{L+d_y}\right)_{\theta_0=90°} - \cos^{-1}\left(\cos\theta_0 + 0.443\frac{\lambda}{L+d_y}\right)_{\theta_0=90°}$

$= \cos^{-1}\left(-0.443\frac{1}{1}\right) - \cos^{-1}\left(0.443\frac{1}{1}\right) = 116.3° - 63.70° = 52.59° = 0.918 \text{ rad}$

Also from Table 6.2

$\Theta_{y0} = 2\left[90° - \cos^{-1}\left(\frac{1.391\lambda}{\pi N d_y}\right)\right] = 2\left[90° - \cos^{-1}\left(\frac{1.391\times 8}{8\pi}\right)\right] = 52.56° = 0.917 \text{ rad}$

Therefore

$$\Theta_h = \left.\sqrt{\frac{1}{\cos^2\theta_0\left[\Theta_{x0}^{-2}\cos^2\phi_0 + \Theta_{y0}^{-2}\sin^2\phi_0\right]}}\right|_{\substack{\phi_0=90°\\ \theta_0=10°}} = \frac{\Theta_{y0}}{\cos\theta_0} = \frac{52.59°}{\cos(10°)} = 53.40° = 0.932 \text{ rad}$$

$$\Psi_h = \left.\sqrt{\frac{1}{\Theta_{x0}^{-2}\sin^2\phi_0 + \Theta_{y0}^{-2}\cos^2\phi_0}}\right|_{\substack{\phi_0=90°\\ \theta_0=10°}} = \Theta_{x0} = 41.52° = 0.7245 \text{ rad}$$

$\Omega_A = \Theta_h \Psi_h = 53.40(41.52) = 2{,}217.17 \text{ (degrees)}^2$

$D_0 \simeq \frac{32{,}400}{\Omega_A \text{ (degrees)}^2} = \frac{32{,}400}{2{,}217.17} = 14.61 = 11.65 \text{ dB}$ and it agrees with the more accurate values above

6-51. $d_x = d_y = \lambda/4$, $N = M = 10$, $\theta_o = 10°$, $\phi_o = 45°$, $R_o = 26\text{dB} = 20$

$$f = 1 + 0.636\left\{\frac{2}{R_o}\cosh\left[\sqrt{(\cosh^{-1}R_o)^2 - \pi^2}\right]\right\}^2$$

$$= 1 + 0.636\left\{\frac{2}{20}\cosh\left[\sqrt{(\cosh^{-1}(20))^2 - \pi^2}\right]\right\}^2 = 1 + 0.636\left\{\frac{2}{20}\cosh\left[\sqrt{(3.69)^2 - \pi^2}\right]\right\}^2$$

$$f = 1 + 0.636\left[\frac{2}{20}\cosh(1.93)\right]^2 = 1 + 0.636\left[\frac{2}{20}(3.52)\right]^2 = 1 + 0.079 = 1.079$$

a. $D = \pi\cos\theta_o\, D_x D_y$, $L_x = L_y = 9\left(\frac{\lambda}{4}\right) = \frac{9}{4}\lambda = 2.25\lambda$

$$D_x = \frac{2R_o^2}{1 + (R_o^2 - 1) f \frac{\lambda}{L_x + d_x}} = D_y = \frac{2R_o^2}{1 + (R_o^2 - 1) f \frac{\lambda}{L_y + d_y}}$$

$$D_x = D_y = \frac{2(20)^2}{1 + (20^2 - 1)\, 1.079\, \frac{1}{2.5}} = 4.62 \Rightarrow 6.65\text{ dB}$$

$$D = \pi \cdot \cos(10°)\,(4.62)(4.62) = 66.04 = 18.20\text{dB}$$

b. For a square array $\Rightarrow \Theta_h = \Theta_{xo}\sec\theta_o = \Theta_{yo}\sec\theta_o$, $\Psi_h = \Theta_{xo} = \Theta_{yo}$

$$\Theta_{xo} = \Theta_{yo} = \left\{\cos^{-1}\left[\cos\theta_o - 0.443\frac{\lambda}{L_x + d_x}\right]_{\theta_o = 90°} - \cos^{-1}\left[\cos\theta_o + 0.443\frac{\lambda}{L_x + d_x}\right]_{\theta_o = 90°}\right\} \cdot f$$

$$= \left\{\cos^{-1}\left[\cos\theta_o - 0.443\frac{\lambda}{L_y + d_y}\right]_{\theta_o = 90°} - \cos^{-1}\left[\cos\theta_o + 0.443\frac{\lambda}{L_y + d_y}\right]_{\theta_o = 90°}\right\} \cdot f$$

$$= \left\{\cos^{-1}\left(0 - 0.443\,\frac{1}{2.5}\right) - \cos^{-1}\left(0 + 0.443\,\frac{1}{2.5}\right)\right\} \cdot (1.079)$$

$\Theta_{xo} = \Theta_{yo} = [100.21 - 79.79]\,1.079 = 20.41\,(1.079) = 22.03° = 0.3844$ rad.

Also from Table 6.2

$$\Theta_{xo} = \Theta_{yo} = 2\left[90° - \cos^{-1}\left(\frac{1.391\lambda}{\pi \times 10 \times \lambda/4}\right)\right](1.079) = 2\,(10.20)(1.079) = 22.01° = 0.3842\text{ rad}$$

Thus

$\Theta_h = \Theta_{xo}\sec\theta_o = \Theta_{yo}\sec\theta_o = 22.03\sec(10°) = 22.03\,(1.015) = 22.37°$
$= 0.390$ rad

$\Psi_h = \Theta_{xo} = \Theta_{yo} = 22.03° = 0.3844$ rad.

c. $\Omega_A = \Theta_h \Psi_h = 22.37\,(22.03) = 492.81\ (\text{degrees})^2 = 0.1501\ (\text{rad.})^2$

6.52 $d_x = d_y = \lambda/8$, $M=10$, $N=8$, $\theta_0 = 10°$, $\phi_0 = 90°$, $R_0 = 30\text{dB} = 31.623$

a. $\beta_x \simeq -k d_x \sin\theta_0 \cos\phi_0 = -\frac{2\pi}{\lambda}\left(\frac{\lambda}{8}\right)\sin(10°)\cos(90°) = 0$

$\beta_y \simeq -k d_y \sin\theta_0 \sin\phi_0 = -\frac{2\pi}{\lambda}\left(\frac{\lambda}{8}\right)\sin(10°)\sin(90°) = -0.1364 \text{ rad} = -7.81°$

b. $D_0 = \pi \cos\theta_0 D_x D_y$

$$f = 1 + 0.636\left\{\frac{2}{31.623}\cosh\left[\sqrt{(\cosh^{-1}(31.623))^2 - \pi^2}\right]\right\}^2$$

$$= 1 + 0.636\left\{\frac{2}{31.623}\cosh\left[\sqrt{(4.147)^2 - \pi^2}\right]\right\}^2 = 1 + 0.636\left[\frac{2}{31.623}(7.523)\right]^2 = 1.144$$

$$D_x = \frac{2R_0^2}{1 + (R_0^2 - 1) f \frac{\lambda}{L_x + d_x}} = \frac{2(31.623)^2}{1 + (31.623^2 - 1)\,1.144\,\frac{1}{1.25}} = 2.185 = 3.395\text{dB}$$

$$D_y = \frac{2R_0^2}{1 + (R_0^2 - 1) f \frac{\lambda}{L_y + d_y}} = \frac{2(31.623)^2}{1 + (31.623^2 - 1)(1.144)\frac{1}{1}} = 1.748 = 2.426\text{dB}$$

$D_0 = \pi \cos(10°)(2.185)(1.748) = 11.817 = 10.725\text{dB}$

c. $\Theta_{x0} = \Theta'_{x0}\big|_{\text{uniform}}\, f = 41.52\,(1.144) = 47.50°$

$\Theta_{y0} = \Theta'_{y0}\big|_{\text{uniform}}\, f = 52.59\,(1.144) = 60.16°$

$\Theta'_{x0}\big|_{\text{uniform}}$ and $\Theta'_{y0}\big|_{\text{uniform}}$ were obtained from the Solution of Problem 6.50

From the Solution of Problem 6.50

$$\Theta_h\Big|_{\substack{\phi_0=90° \\ \theta_0=10°}} = \frac{\Theta_{y0}}{\cos\theta_0} = \frac{60.16°}{\cos(10°)} = 1.0154\,(60.16°) = 61.09° = 1.066 \text{ rad}$$

$$\Psi_h\Big|_{\substack{\phi_0=90° \\ \theta_0=10°}} = \Theta_{x0} = 47.50° = 0.829 \text{ rad}$$

$\Omega_A = \Theta_h \Psi_h = 61.09\,(47.50) = 2{,}901.775 (\text{degrees})^2 = 0.884\,(\text{rad})^2$

$D_0 \simeq \frac{32{,}400}{\Omega_A (\text{degrees})^2} = \frac{32{,}400}{2{,}901.775} = 11.17 = 10.48\text{dB}$ and it agrees closely with the more accurate value of 11.817 or 10.725dB

6-53. In the design of an array, the maximum occurs at $\theta = \theta_0$ at the design frequency $f = f_0$ which has been used to determine the progressive phase between the elements. As the shifts from $f_0$, the maximum also shifts to some other angle $\theta$ array from $\theta_0$.

At a frequency $f_h$, the maximum of the array factor will be 0.707 of the normalized maximum value of unity. The frequency $f_h$ is referred to as the half-power frequency, and it is used to determine the frequency bandwidth over which the pattern varies over an amplitude of 3-dB from the maximum at $f_0$. To determine the frequency $f_h$ and the 3-dB frequency bandwidth, the normalized array factor of (6-10c) is written using (6-21) as

$$AF = \frac{1}{N} \frac{\sin\left[\frac{N\pi d}{\lambda}(\cos\theta - \cos\theta_0)\right]}{\sin\left[\frac{\pi d}{\lambda}(\cos\theta - \cos\theta_0)\right]}$$

The frequency is obscured in the array factor. To be valid over a band of frequencies, the wavelengths $\lambda$ and $\lambda_0$ and the frequencies $f$ and $f_0$ should be shown explicitly. Using the relation $v = \lambda f$, the array factor can be written as

$$AF = \frac{1}{N} \frac{\sin\left[\frac{\pi N d}{v}(f\cos\theta - f_0\cos\theta_0)\right]}{\sin\left[\frac{\pi d}{v}(f\cos\theta - f_0\cos\theta_0)\right]}$$

which peaks at $\theta = \theta_0$ when $f = f_0$. At any other frequency, the array factor peaks when

$$f\cos\theta - f_0\cos\theta_0 = 0 \Rightarrow \cos\theta = \frac{f_0}{f}\cos\theta_0$$

The half-power of the array factor is obtained by letting $\theta = \theta_0$ and occurs

$$\frac{N\pi d}{v}(f_h\cos\theta_0 - f_0\cos\theta_0) = \frac{N\pi d}{v}\cos\theta_0(f_h - f_0) = 1.391$$

or

$$(f_h - f_0) = \frac{1.391v}{N\pi d\cos\theta} = \frac{0.886v}{Nd\cos\theta_0} = \frac{0.886v}{(L+d)\cos\theta}$$

(Continued)

6-53 (Cont'd)

Therefore the 3-dB frequency bandwidth is

$$BW(3\text{-dB}) = \frac{0.886v}{Nd\cos\theta_0} = \frac{0.886v}{(L+d)\cos\theta_0}$$

Therefore the bandwidth of an array depends not on the frequency operation but rather on the array length and scan angle.

This is a fundamental constraint on wide-instantaneous bandwidth of arrays.

7-1. Using (7-4), the array factor can be written as

$$AF = a_1 + a_2 z + a_3 z^2 = 1 + 2z + z^2 = (1+z)^2$$

which has two roots and both occur at $z = -1$.

a. The nulls of the array factor can be found by setting $z$ equal to $-1$. Thus

$$z = e^{j(kd\cos\theta + \beta)} = -1 \Rightarrow kd\cos\theta + \beta = \pm(2n+1)\pi, \quad n = 0, 1, 2, \cdots$$

For $d = \lambda/4$

$$kd\cos\theta + \beta = \frac{2\pi}{\lambda}\left(\frac{\lambda}{4}\right)\cos\theta + \beta = \frac{\pi}{2}\cos\theta + \beta = \pm(2n+1)\pi, \quad n = 0, 1, 2, \cdots$$

$$\theta = \cos^{-1}\left[\frac{2}{\pi}(-\beta + (2n+1)\pi)\right], \quad n = 0, 1, 2, \cdots$$

For $\beta = 0$;

$$\theta = \cos^{-1}[\pm 2(2n+1)] = \text{does not exist, } \underline{\text{No zeros}}.$$

For $\beta = \frac{\pi}{2}$;

$$\theta = \cos^{-1}[2(-0.5 \pm (2n+1))]$$

For $n = 0 \Rightarrow \theta = \cos^{-1}[2(-0.5 \pm 1)]$

Using the positive sign between the two terms

$$\theta = \cos^{-1}(+1) = 0^\circ$$

For $\beta = \pi$;

$$\theta = \cos^{-1}[2(-1 \pm (2n+1))]$$

For $n = 0 \Rightarrow \theta = \cos^{-1}[2(-1 \pm 1)] = \cos^{-1}(0) = 90^\circ$

For $\beta = 3\pi/2$;

$$\theta = \cos^{-1}[2(-1.5 \pm (2n+1))]$$

For $n = 0 \Rightarrow \theta = \cos^{-1}[2(-1.5 \pm 1)] = \cos^{-1}(-1) = 180^\circ$

The computed patterns for $d = \frac{\lambda}{4}$ and $\beta = 0, \pi/2, \pi, 3\pi/2$ are shown plotted in Fig. P7.1. The nulls do occur at the computed angles.

(Continued)

7-1. (Cont'd)

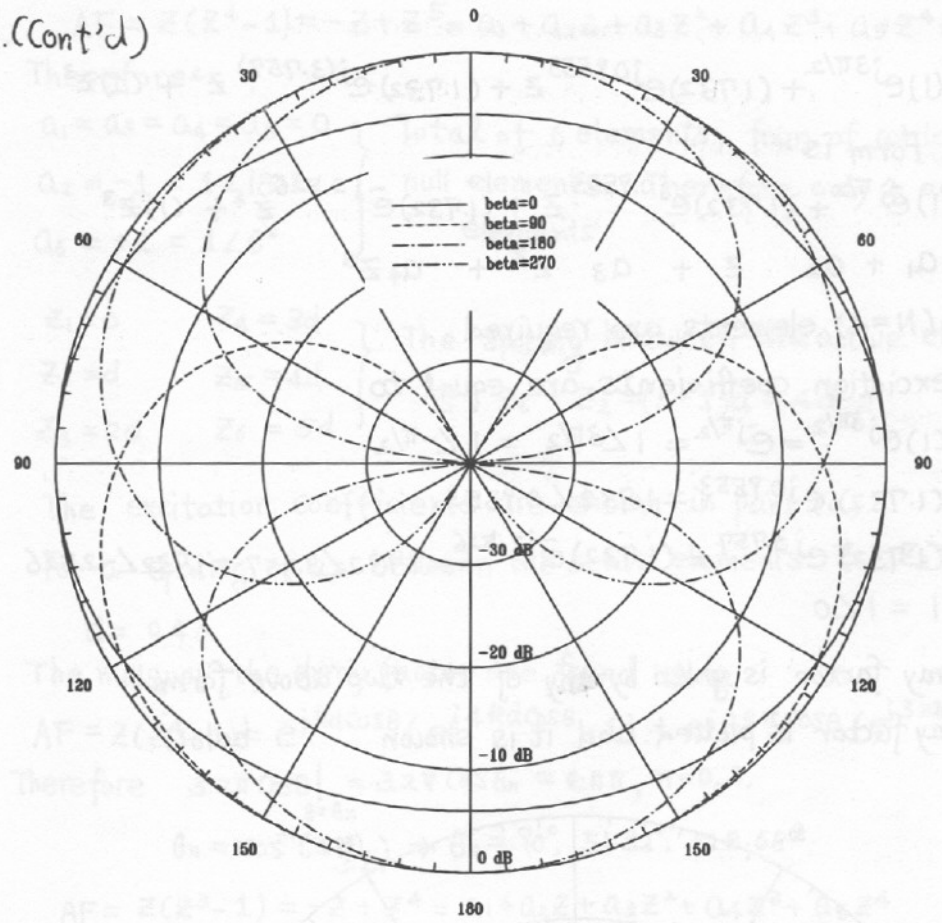



Fig. P7.1

7-2. For $\beta = 0$ and $d = \lambda/4$

$$\psi = kd\cos\theta + \beta = \frac{2\pi}{\lambda}\left(\frac{\lambda}{4}\right)\cos\theta = \frac{\pi}{2}\cos\theta$$

which reduces to

$\theta = 0°$: $\psi = \psi_1 = \frac{\pi}{2} \Rightarrow z_1 = j$

$\theta = 60°$: $\psi = \psi_2 = \frac{\pi}{2}\left(\frac{1}{2}\right) = \frac{\pi}{4} \Rightarrow z_2 = (1+j)/\sqrt{2}$

$\theta = 120°$: $\psi = \psi_3 = \frac{\pi}{2}\left(-\frac{1}{2}\right) = -\frac{\pi}{4} \Rightarrow z_3 = (1-j)/\sqrt{2}$

Therefore the array factor of (7-5) can be written as

$$AF = (z - z_1)(z - z_2)(z - z_3) = (z - j)\left(z - \frac{1}{\sqrt{2}}(1+j)\right)\left(z - \frac{1}{\sqrt{2}}(1-j)\right)$$

$$= -j + (1 + j\sqrt{2})z - (\sqrt{2} + j)z^2 + z^3$$

(Continued)

7-2 (Cont'd)

$$AF = (1)e^{j3\pi/2} + (1.732)e^{j0.9553} \cdot z + (1.732)e^{j(3.757)} \cdot z^2 + (1)z^3$$

Another Form is

$$AF = (1)e^{-j\pi/2} + (1.732)e^{j0.9553} \cdot z + (1.732)e^{-j2.526} z^2 + (1)z^3$$

$$AF = a_1 + a_2 \quad z + a_3 \quad z^2 + \quad a_4 z^3$$

a. Four ($N=4$) elements are required.

b. The exciation coefficients are equal to

$$a_1 = (1)e^{j3\pi/2} = e^{j\pi/2} = 1\angle 3\pi/2 = 1\angle -\pi/2$$

$$a_2 = (1.732)\, e^{j0.9553} = 1.732\angle 0.9553$$

$$a_3 = (1.732)\, e^{j3.757} = (1.732)\, e^{-j2.526} = 1.732\angle 3.757 = 1.732\angle -2.526$$

$$a_4 = 1 = 1\angle 0$$

c. The array factor is given by any of the two above forms.

d. The array factor is plotted and it is shown below.

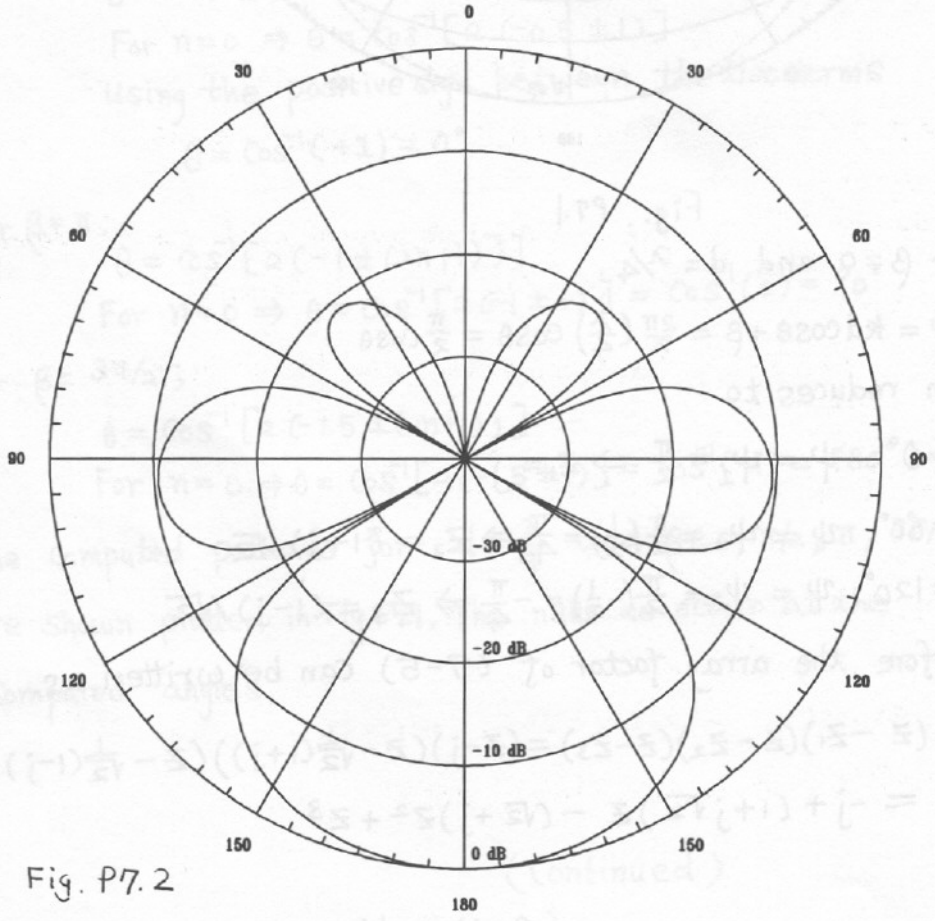



Fig. P7.2

7-3. $AF = z(z^4-1) = -z+z^5 = a_1 + a_2 z + a_3 z^2 + a_4 z^3 + a_5 z^4 + a_6 z^5$

Therefore

a. $a_1 = a_3 = a_4 = a_5 = 0$
$a_2 = -1 = 1\angle 180°$
$a_6 = +1 = 1\angle 0°$
} Total of 6 elements, four of which are null elements. Therefore only 2 active elements.

b. $z_1 = 0$ $z_4 = 3d$
$z_2 = d$ $z_5 = 4d$
$z_3 = 2d$ $z_6 = 5d$
} The spacing between the active elements is $z_6 - z_2 = (5-1)d = 4d$

c. The excitation coefficients are shown in part (a).

d. For a spacing of $2\lambda$ between the active elements $z_6 - z_2 = 4d = 1.6\lambda$

$d = 0.4\lambda$

The nulls of the array factor are found using

$AF = z(z^4-1) = e^{jkd\cos\theta}(e^{j4kd\cos\theta} - 1) = e^{j0.8\pi\cos\theta}(e^{j3.2\pi\cos\theta} - 1) = 0$

Therefore $3.2\pi\cos\theta\big|_{\theta=\theta_n} = 3.2\pi\cos\theta_n = \pm n\pi$, $n = 0, 2.$

$\theta_n = \cos^{-1}(\pm\frac{n}{3.2}) \Rightarrow \theta_n = 90°, 51.32°, 128.68°.$

7-4. $AF = z(z^3-1) = -z + z^4 = a_1 + a_2 z + a_3 z^2 + a_4 z^3 + a_5 z^4$

Therefore

a. $a_1 = a_3 = a_4 = 0$
$a_2 = -1 = 1\angle 180°$
$a_5 = 1 = 1\angle 0°$
} Total of 5 elements, three of which are null elements. Therefore only 2 active elements.

b. $z_1 = 0$ $z_4 = 3\cdot d$
$z_2 = d$ $z_5 = 4\cdot d$
$z_3 = d\cdot 2$
} The spacing between the active elements is $z_5 - z_2 = (4-1)d = 3d$

c. The exciation coefficients are shown in part (a)

d. $z_5 - z_2 = 1.5\lambda$, $d = 0.5\lambda$

The nulls of the array factor are found using

$AF = z(z^3-1) = e^{j\pi\cos\theta}(e^{j3\pi\cos\theta} - 1) = 0.$

$3\pi\cos\theta\big|_{\theta=\theta_n} = 3\pi\cos\theta_n = \pm n\pi$, $n = 0, 2.$

$\theta_n = \cos^{-1}(\pm\frac{n}{3}) \Rightarrow \theta_n = 90°, 48.19°, 131.81°$

7-5. $SF(\theta) = \begin{cases} 1 & 40^\circ \le \theta \le 140^\circ \\ 0 & \text{elsewhere} \end{cases}$

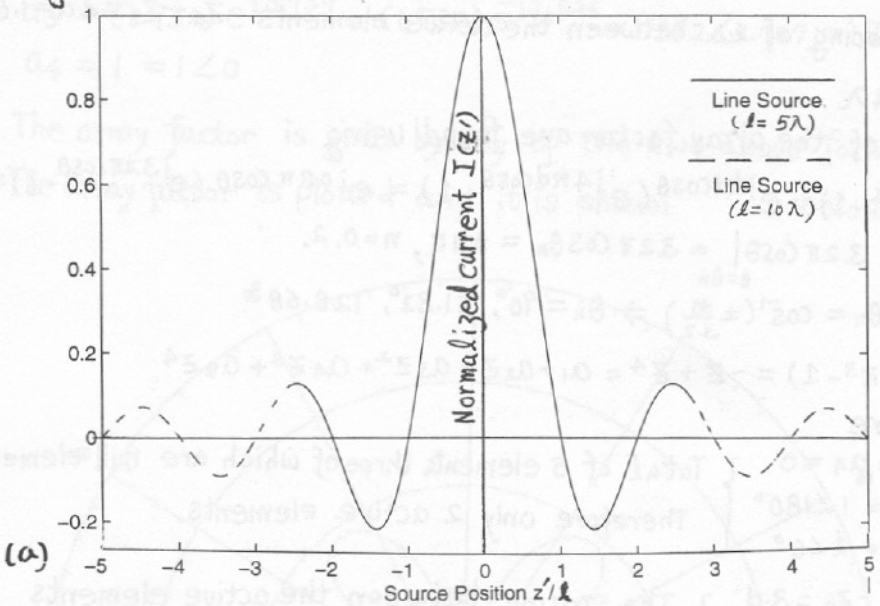
$40^\circ \le \theta \le 140^\circ \Rightarrow \frac{\pi}{4.5} \le \theta \le \frac{7}{9}\pi$

Using Equation (7-8a), $-k/(1.3054) \le \xi \le k/(1.3054)$ for $kz = 0$, and $140^\circ \ge \theta \ge 40^\circ$. Thus

$$I(z') = \frac{k}{\pi(1.3054)}\left[\sin\left(\frac{kz'}{1.3054}\right) \Big/ \left(\frac{kz'}{1.3054}\right)\right] \quad \text{and}$$

$$SF(\theta)_d \simeq SF(\theta)_a = \frac{1}{\pi}\left\{Si\left[\frac{l}{\lambda}\pi\left(\cos\theta + \frac{1}{1.3054}\right)\right] - Si\left[\frac{l}{\lambda}\pi\left(\cos\theta - \frac{1}{1.3054}\right)\right]\right\}$$

For $l = 5\lambda$ and $l = 10\lambda$, the normalized current distribution is shown in Fig. P7.5(a) and the patterns in Figure P7.5(b).



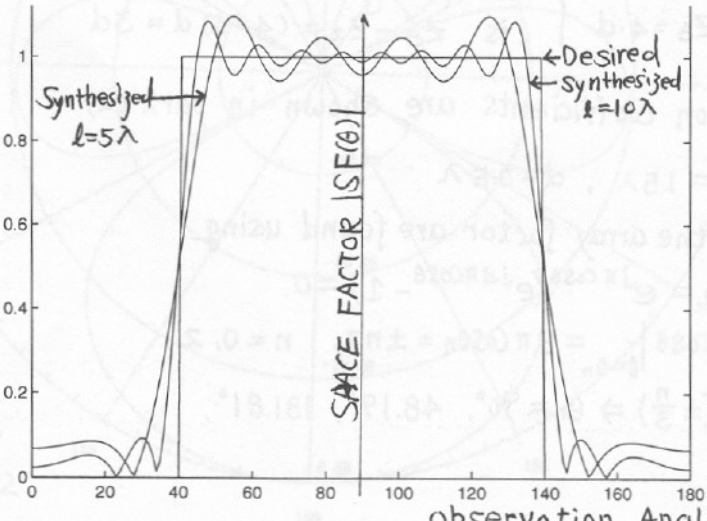




Fig. P7-5. (b)

7-6. Using Equation (7-8a), $-k/2 \le \xi \le k/2$ for $k_z = 0$ and $120° \ge \theta \ge 60°$. Thus

$$I(z') = \frac{k}{2\pi}\left[\sin\left(\frac{kz'}{2}\right)/\left(\frac{kz'}{2}\right)\right]$$

and $$SF(\theta)_d \simeq SF(\theta)_a = \frac{1}{\pi}\left\{Si\left[\frac{\ell}{\lambda}\pi(\cos\theta + \tfrac{1}{2})\right] - Si\left[\frac{\ell}{\lambda}\pi(\cos\theta - \tfrac{1}{2})\right]\right\}$$

For $\ell = 5\lambda$ and $\ell = 10\lambda$, the normalized current distribution is shown in Fig. P7.6 (a) and the patterns in Figure P7.6 (b).

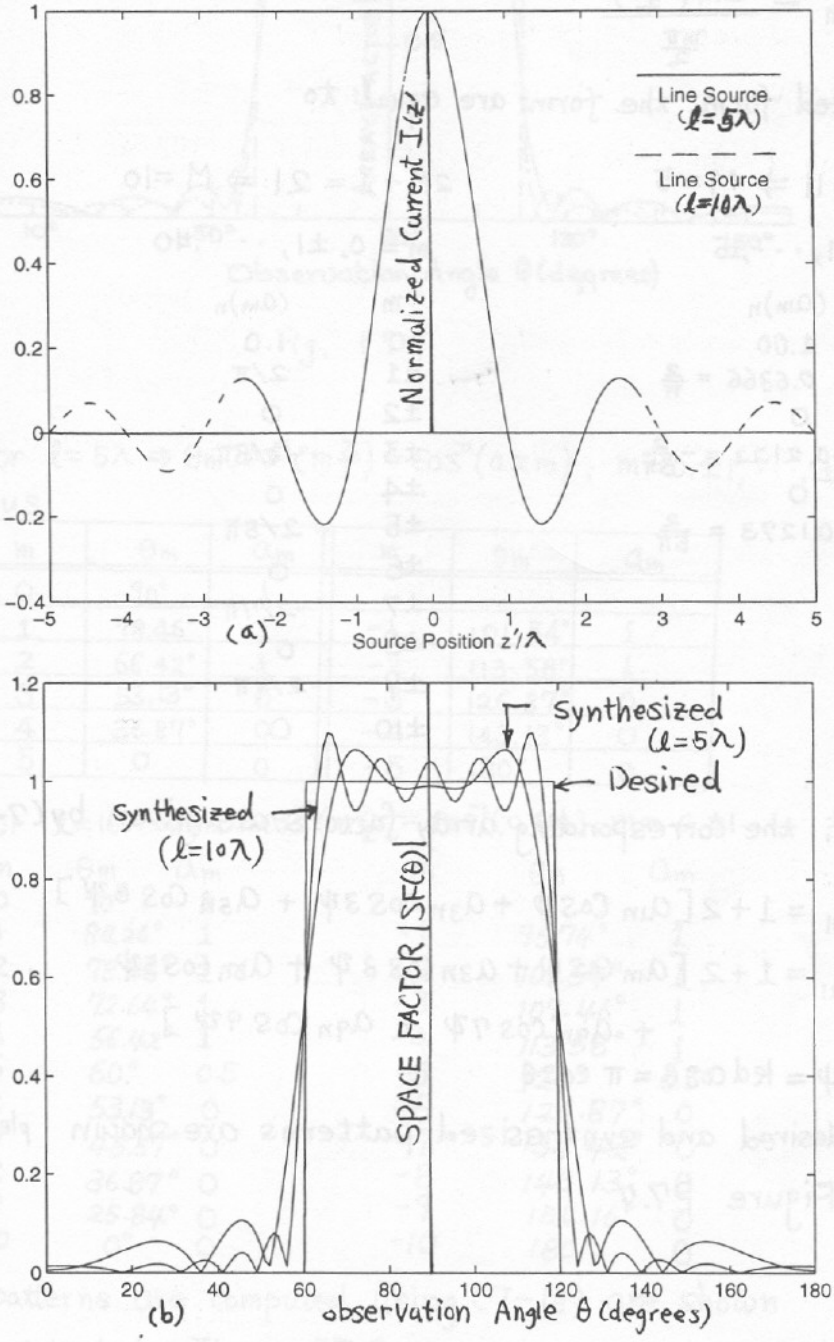



Fig. P7-6

7-7 For $60° \leq \theta \leq 120°$, the excitation coefficients can be obtained using

$$a_m = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{jm\psi}\, d\psi = \frac{1}{2}\left[\frac{\sin\left(\frac{m\pi}{2}\right)}{\frac{m\pi}{2}}\right]$$

or in normalized form by

$$(a_m)_n = \frac{\sin\left(\frac{m\pi}{2}\right)}{\frac{m\pi}{2}}$$

In normalized form, the forms are equal to

$2M+1 = 11 \Rightarrow M = 5$

$m = 0, \pm 1, \cdots, \pm 5$

| $m$ | $(a_m)_n$ |
|---|---|
| 0 | 1.00 |
| $\pm 1$ | $0.6366 = \frac{2}{\pi}$ |
| $\pm 2$ | 0 |
| $\pm 3$ | $-0.2122 = -\frac{2}{3\pi}$ |
| $\pm 4$ | 0 |
| $\pm 5$ | $0.1273 = \frac{2}{5\pi}$ |

$2M+1 = 21 \Rightarrow M = 10$

$m = 0, \pm 1, \cdots, \pm 10$

| $m$ | $(a_m)_n$ |
|---|---|
| 0 | 1.0 |
| $\pm 1$ | $2/\pi$ |
| $\pm 2$ | 0 |
| $\pm 3$ | $-2/3\pi$ |
| $\pm 4$ | 0 |
| $\pm 5$ | $2/5\pi$ |
| $\pm 6$ | 0 |
| $\pm 7$ | $-2/7\pi$ |
| $\pm 8$ | 0 |
| $\pm 9$ | $2/9\pi$ |
| $\pm 10$ | 0 |

For $d = \lambda/2$, the corresponding array factors are given by (7-3a), or

$$AF(\theta)_{11} = 1 + 2\left[a_{1n}\cos\psi + a_{3n}\cos 3\psi + a_{5n}\cos 5\psi\right]$$

$$AF(\theta)_{21} = 1 + 2\left[a_{1n}\cos\psi + a_{3n}\cos 3\psi + a_{5n}\cos 5\psi + a_{7n}\cos 7\psi + a_{9n}\cos 9\psi\right]$$

where $\psi = kd\cos\theta = \pi\cos\theta$

The desired and synthesized patterns are shown plotted in Figure P7.7

(Continued)

7-7 (Cont'd)

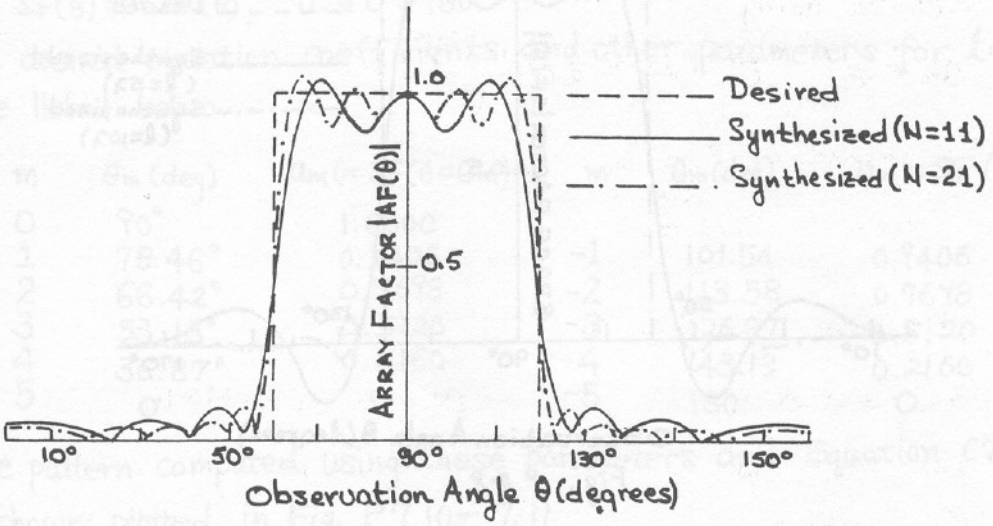



Fig. P7.7

7-8. For $\ell = 5\lambda \Rightarrow \theta_m = \cos^{-1}(m\frac{\lambda}{\ell}) = \cos^{-1}(0.2m)$, $m = 0, \pm 1, \cdots, \pm 5$.

Thus

| m | $\theta_m$ | $a_m$ | m | $\theta_m$ | $a_m$ |
|---|---|---|---|---|---|
| 0 | 90° | 1 | | | |
| 1 | 78.46° | 1 | -1 | 101.54° | 1 |
| 2 | 66.42° | 1 | -2 | 113.58° | 1 |
| 3 | 53.13° | 0 | -3 | 126.87° | 0 |
| 4 | 36.87° | 0 | -4 | 143.13° | 0 |
| 5 | 0 | 0 | -5 | 180° | 0 |

For $\ell = 10\lambda \Rightarrow \theta_m = \cos^{-1}(m\frac{\lambda}{\ell}) = \cos^{-1}(0.1m)$, $m = 0, \pm 1, \cdots, \pm 10$, Thus

| m | $\theta_m$ | $a_m$ | m | $\theta_m$ | $a_m$ |
|---|---|---|---|---|---|
| 0 | 90° | 1 | | | |
| 1 | 84.26° | 1 | -1 | 95.74° | 1 |
| 2 | 78.46° | 1 | -2 | 101.54° | 1 |
| 3 | 72.64° | 1 | -3 | 107.46° | 1 |
| 4 | 66.42° | 1 | -4 | 113.58° | 1 |
| 5 | 60° | 0.5 | -5 | 120° | 0.5 |
| 6 | 53.13° | 0 | -6 | 126.87° | 0 |
| 7 | 45.57° | 0 | -7 | 134.43° | 0 |
| 8 | 36.87° | 0 | -8 | 143.13° | 0 |
| 9 | 25.84° | 0 | -9 | 154.16° | 0 |
| 10 | 0° | 0 | -10 | 180° | 0 |

The patterns are computed using (7-18) are shown plotted in Figure P7.8

(Continued)

7-8 (Cont'd)

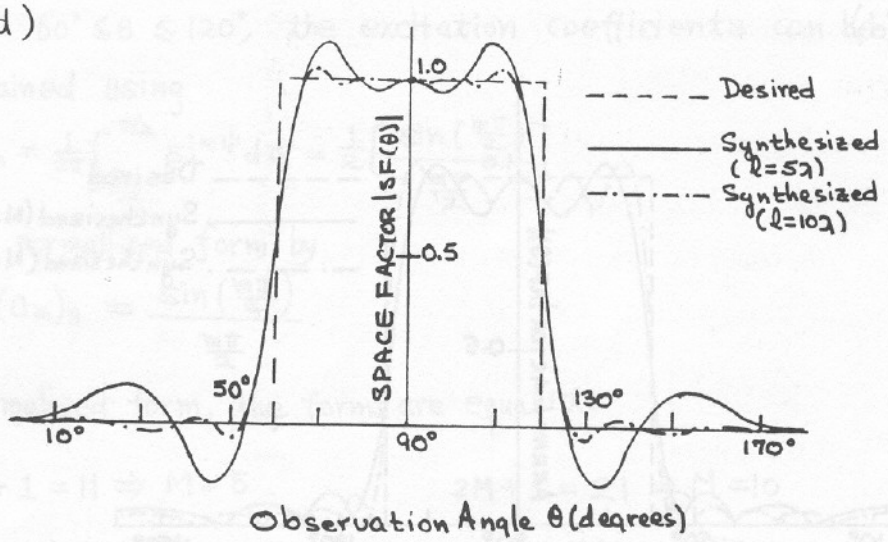



Fig. P 7-8

7-9. Since the number of elements ($N=11$ and $N=21$) is identical, respectively, to the number of samples for $\ell=5\lambda$ and $\ell=10\lambda$ in Problem 7-8 when $d=\lambda/2$, the corresponding exciation coefficients and the angles where each of the composing functions is maximum are identically the same as those tabulated in the solution of Prob. 7-8. The patterns computed using these parameters, along with Equation (7-21), are shown plotted in Fig. P7.9.

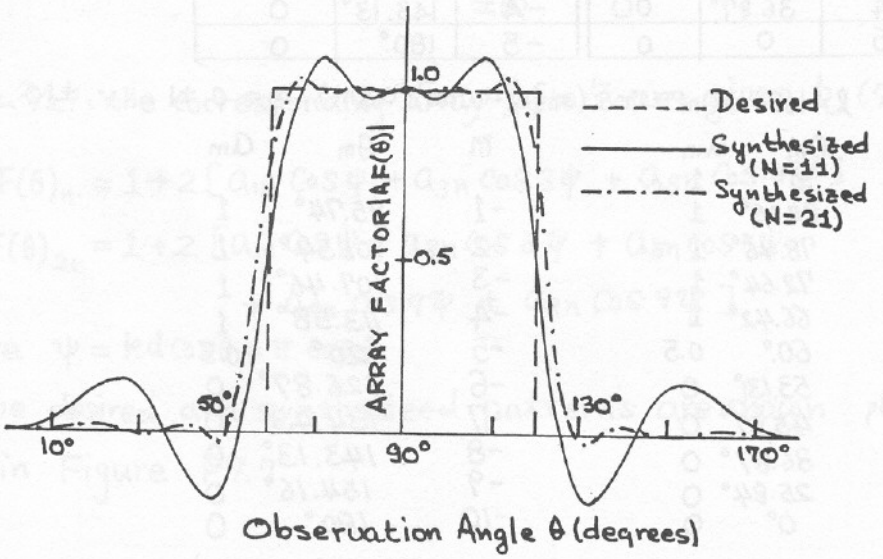



Fig. P7.9

7-10. For a desired pattern of

$SF(\theta) = \sin^3(\theta), \quad 0^\circ \leq \theta \leq 180^\circ$

the desired exciation coefficients and other parameters for $\ell = 5\lambda$ are listed below.

| m | $\theta_m$ (deg.) | $a_m (= SF(\theta = \theta_m)$ | m | $\theta_m$ (deg.) | $a_m = SF(\theta = \theta_m)$ |
|---|---|---|---|---|---|
| 0 | 90° | 1.0000 | | | |
| 1 | 78.46° | 0.9406 | -1 | 101.54 | 0.9406 |
| 2 | 66.42° | 0.7698 | -2 | 113.58 | 0.7698 |
| 3 | 53.13° | 0.5120 | -3 | 126.87 | 0.5120 |
| 4 | 36.87° | 0.2160 | -4 | 143.13 | 0.2160 |
| 5 | 0 | 0 | -5 | 180. | 0 |

The pattern computed using these parameters and Equation (7-18) is shown plotted in Fig. P7.10-7.11.

It is identical to that obtained using discrete array modeling.

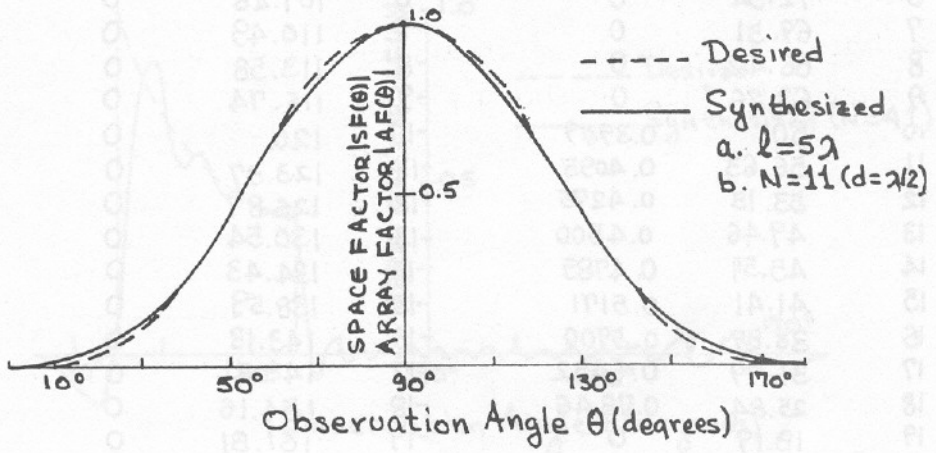



Fig. P7.10-7.11

7-11. Since the number of elements $N = 11$ for $d = \lambda/2$ is identical as the number of samples of Prob. 7.10, the excitation coefficients and other required parameters are the same as those listed in the Solution of Prob. 7-10. Using these and equation (7-21), the pattern computed is shown plotted in Fig. P7.10 – 7.11 It is identical to that obtained using continuous-source modeling.

7-12 Since the desired pattern is given by

$$\Rightarrow SF(\theta) = \begin{cases} 0.342\ \csc(\theta) & 20^\circ \leq \theta \leq 60^\circ \\ 0 & \text{Elsewhere} \end{cases}$$

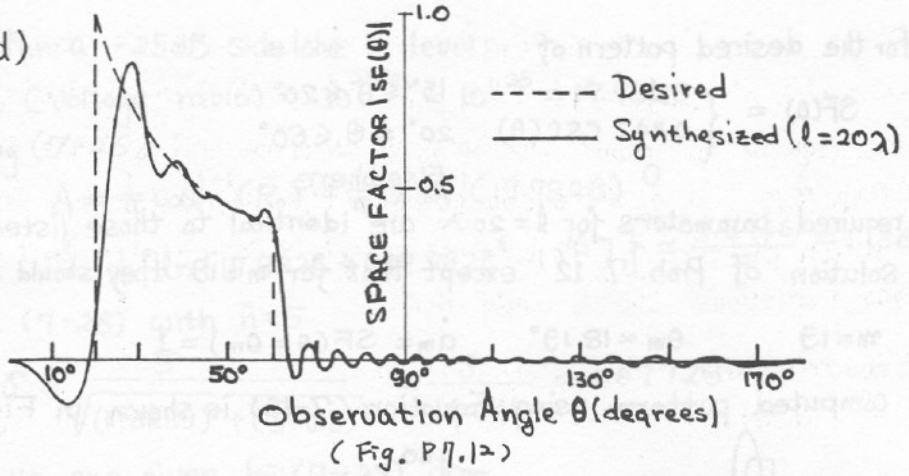
the required parameters for a Woodward line-source design of $\ell = 20\lambda$ are listed below

| m | $\theta_m$ (deg.) | $a_m = SF(\theta = \theta_m)$ | m | $\theta_m$ (deg.) | $a_m = SF(\theta = \theta_m)$ |
|---|---|---|---|---|---|
| 0 | 90. | 0 | | | |
| 1 | 87.13 | 0 | -1 | 92.87 | 0 |
| 2 | 84.26 | 0 | -2 | 95.74 | 0 |
| 3 | 81.37 | 0 | -3 | 98.63 | 0 |
| 4 | 78.46 | 0 | -4 | 101.54 | 0 |
| 5 | 75.52 | 0 | -5 | 104.48 | 0 |
| 6 | 72.54 | 0 | -6 | 107.46 | 0 |
| 7 | 69.51 | 0 | -7 | 110.49 | 0 |
| 8 | 66.42 | 0 | -8 | 113.58 | 0 |
| 9 | 63.26 | 0 | -9 | 116.74 | 0 |
| 10 | 60. | 0.3949 | -10 | 120. | 0 |
| 11 | 56.63 | 0.4095 | -11 | 123.37 | 0 |
| 12 | 53.13 | 0.4275 | -12 | 126.87 | 0 |
| 13 | 49.46 | 0.4500 | -13 | 130.54 | 0 |
| 14 | 45.57 | 0.4789 | -14 | 134.43 | 0 |
| 15 | 41.41 | 0.5171 | -15 | 138.59 | 0 |
| 16 | 36.87 | 0.5700 | -16 | 143.13 | 0 |
| 17 | 31.79 | 0.6492 | -17 | 148.21 | 0 |
| 18 | 25.84 | 0.7846 | -18 | 154.16 | 0 |
| 19 | 18.19 | 0 | -19 | 161.81 | 0 |
| 20 | 0 | 0 | -20 | 180 | 0 |

The pattern computed using these parameters and Equation (7-18) is shown plotted in Fig. P7.12.

(Continued)

7-12 (Cont'd)




(Fig. P7.12)

7-13. Since the number of elements N=41 with a spacing of $d = \lambda/2$ is identical to the number of samples in Prob. 7.12, the excitation coefficients and other parameters are identical to those listed in the solution of Prob 7.12. Using these parameters along with Equation (7-21), the pattern computed is shown plotted in Fig. P7.13

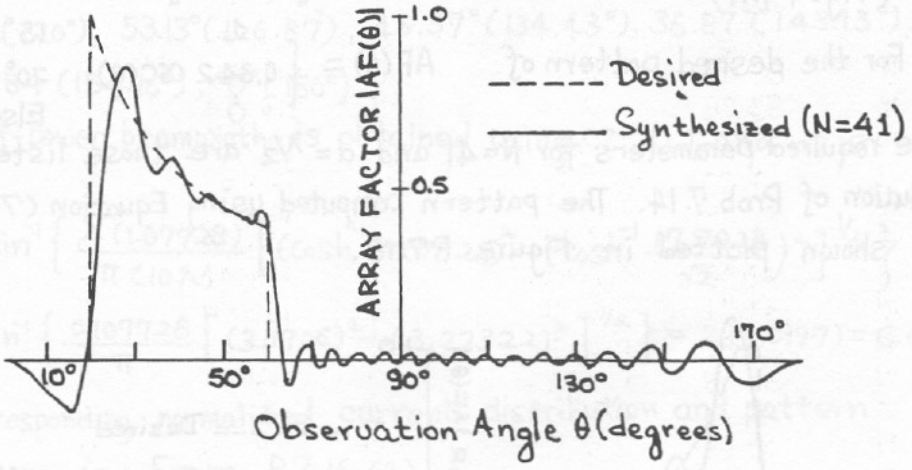



Fig. P7.13

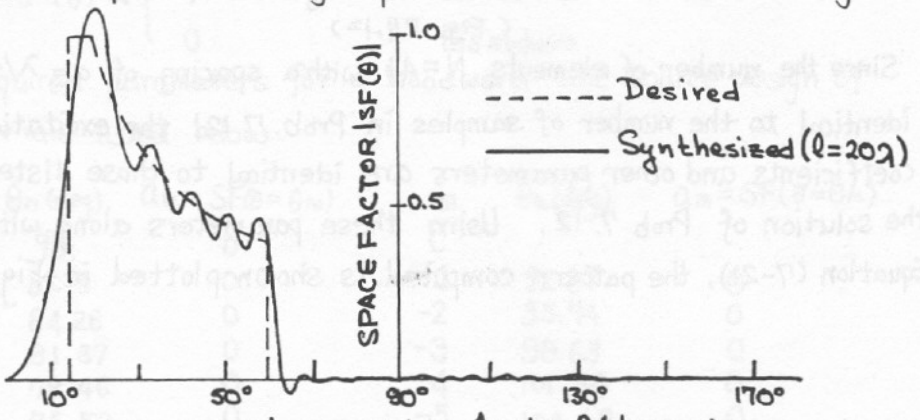
7-14 For the desired pattern of

$$SF(\theta) = \begin{cases} 1 & 15^\circ \leq \theta \leq 20^\circ \\ 0.342\,CSC(\theta) & 20^\circ \leq \theta \leq 60^\circ \\ 0 & \text{Elsewhere} \end{cases}$$

the required parameters for $\ell = 20\lambda$ are identical to those listed in the solution of Prob. 7.12 except that for $m = 19$ they should be equal to

$m = 19 \qquad \theta_m = 18.19^\circ \qquad a_m = SF(\theta = \theta_m) = 1$

The computed pattern using Equation (7-18) is shown in Fig. P7-14.




(Fig. P7.14)

7-15. For the desired pattern of

$$AF(\theta) = \begin{cases} 1 & 15^\circ \leq \theta \leq 20^\circ \\ 0.342\,CSC(\theta) & 20^\circ \leq \theta \leq 60^\circ \\ 0 & \text{Elsewhere} \end{cases}$$

the required parameters for $N = 41$ and $d = \lambda/2$ are those listed in the solution of Prob. 7.14. The pattern computed using Equation (7-21) is shown plotted in Figure P7.15.

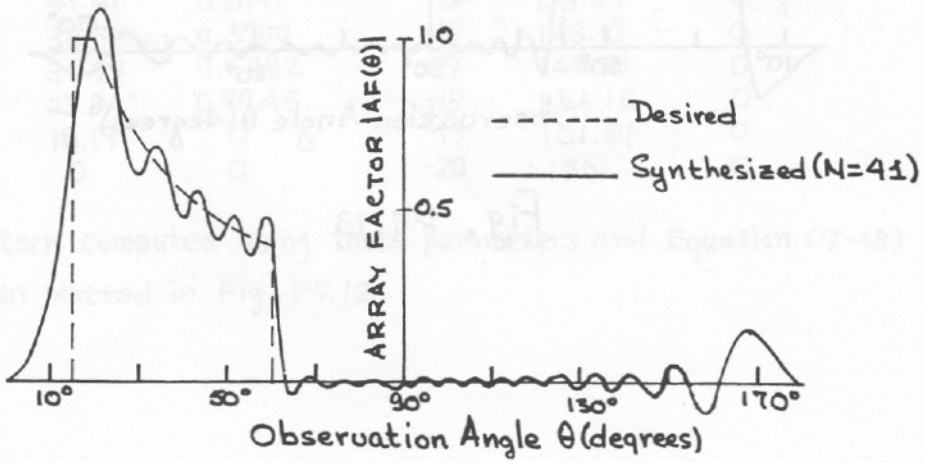



Fig. P7.15

7-16. a. For a -25dB side lobe level

$R_o$ (voltage ratio) $= 10^{25/20} = 10^{1.25} = 17.7828$

Using (7-26)

$$A = \frac{1}{\pi}\cosh^{-1}(R_o) = \frac{1}{\pi}\cosh^{-1}(17.7828)$$

$$= \frac{1}{\pi}\left\{\ln\left[17.7828 + (17.7828^2 - 1)^{1/2}\right]\right\} = \frac{3.5706}{\pi} = 1.13655$$

and (7-28) with $\bar{n} = 5$

$$\sigma = \frac{5}{\sqrt{(1.13655)^2 + (5-0.5)^2}} = \frac{5}{4.6413} = 1.07728$$

The nulls are given by (7-29) or

$$V_n = u_n/\pi = \begin{cases} \pm\sigma\sqrt{A^2 + (n-0.5)^2} & 1 \le n < 5 \\ \pm\pi & 5 \le n \le \infty \end{cases}$$

$V_n = \pm 1.3376, \pm 2.0274, \pm 2.9585, \pm 3.9643, \pm 5, \pm 6, \pm 7,$

$\pm 8, \pm 9, \pm 10, \pm 11, \pm 12, \ldots$

For $\ell = 10\lambda$, the corresponding angles are given by

$\theta_n = 82.31°(97.68°), 78.30°(101.70°), 72.79°(107.21°), 66.64°(113.36°),$
$60°(120°), 53.13°(126.87°), 45.57°(134.43°), 36.87°(143.13°),$
$25.84(154.16°), 0°(180°)$

The half-power beamwidth is obtained using (7-31), and it is equal to

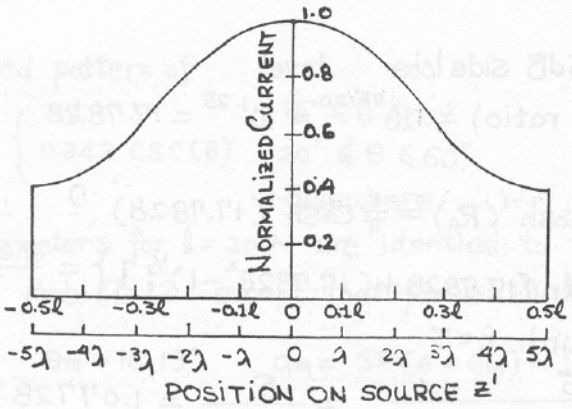
$$\Theta_0 \simeq 2\sin^{-1}\left\{\frac{\lambda(1.07728)}{\pi(10\lambda)}\left[(\cosh^{-1}(17.7828))^2 - (\cosh^{-1}\frac{17.7828}{\sqrt{2}})^2\right]^{1/2}\right\}$$

$$\simeq 2\sin^{-1}\left\{\frac{0.107728}{\pi}\left[(3.5706)^2 - (3.22322)^2\right]^{1/2}\right\} = 2(3.0197) = 6.039°$$

The corresponding normalized current distribution and pattern are shown in Figure P7.16 (a).

(Continued)

7-16 (Cont'd)



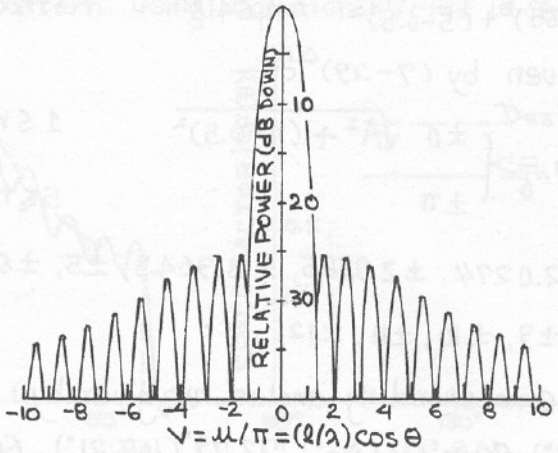




Fig. P7-16(a)

b. For a -20 dB side lobe level

$R_0$ (voltage ratio) $= 10^{20/20} = 10$

Using (7-26) $A = \frac{1}{\pi}\cosh^{-1}(10) = \frac{2.9932}{\pi} = 0.95277$

and (7-28) with $\bar{n} = 10$

$$\sigma = \frac{10}{\sqrt{(0.95277)^2 + (10-0.5)^2}} = \frac{10}{9.5476} = 1.0474$$

The nulls are given by (7-29) or

$V_n = \pm 1.1270, \pm 1.8612, \pm 2.8020, \pm 3.7992, \pm 4.8176, \pm 5.8464,$
$\pm 6.8807, \pm 7.9185, \pm 8.9585, \pm 10, \pm 11, \pm 12, \cdots$

For $\ell = 10\lambda$, the corresponding angles are given by

$\theta_n = 83.53°(96.47°), 79.27°(100.73°), 73.73°(106.27°), 67.67°(112.33°),$
$61.20°(118.80°), 54.22°(125.78°), 46.52°(133.48°),$
$37.64°(142.36°), 26.38°(153.62°), 0°(180°)$

(Continued)

7-16 b. (Cont'd) The half-power beamwidth is obtained using (7-31), and it is equal to

$$\Theta_0 \simeq 2\sin^{-1}\left\{\frac{\lambda(1.0474)}{\pi(10)}\left[(\cosh^{-1}10)^2-(\cosh^{-1}\tfrac{10}{\sqrt{2}})^2\right]^{1/2}\right\}$$

$$\simeq 2\sin^{-1}\left\{\frac{0.10474}{\pi}\left[(2.99322)^2-(2.64412)^2\right]^{1/2}\right\} = 2(2.68) = 5.36^\circ$$

The corresponding normalized current distribution and pattern are shown plotted in Fig. P7.16 (b).

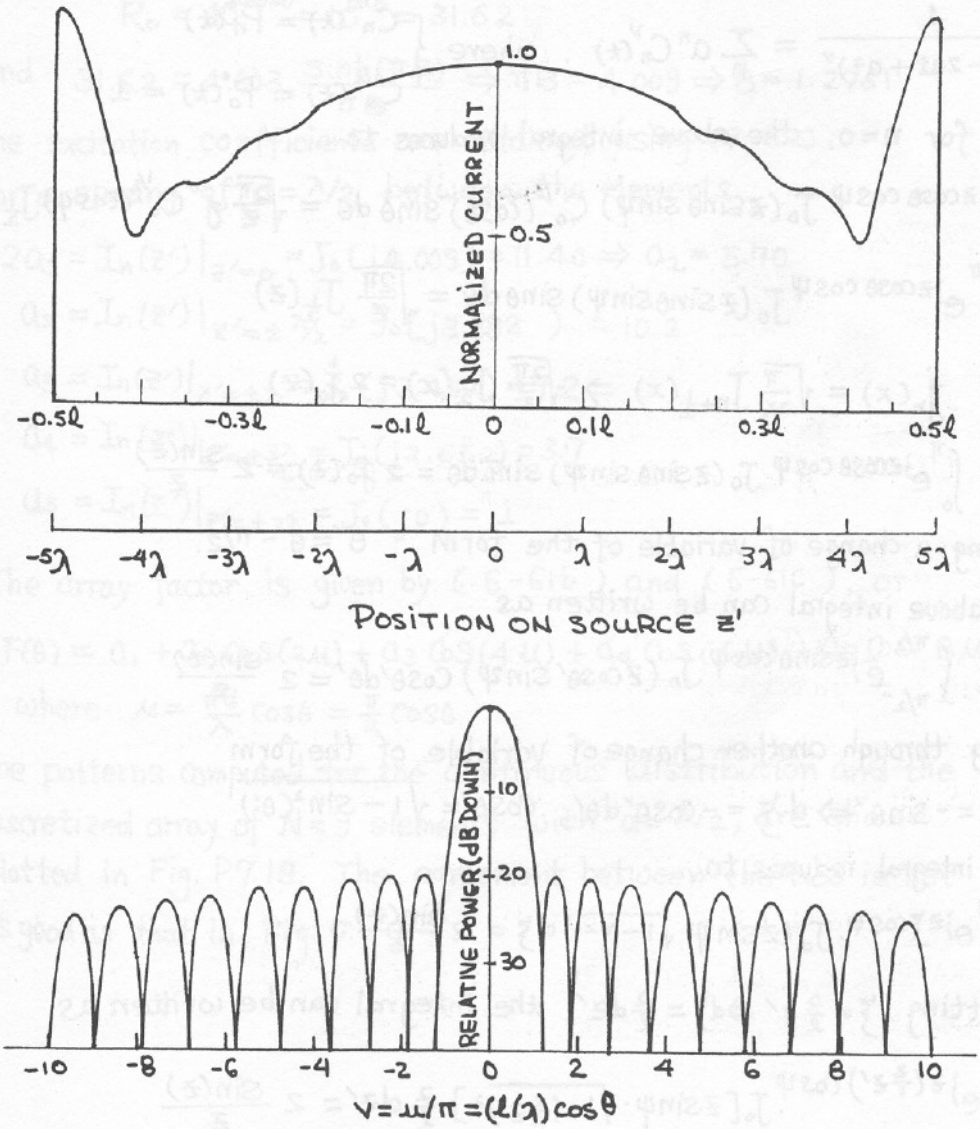






Fig. P7-16 b.

(Continued)

7-17. From Page 379 of Reference 12 (Formula 1)

$$\int_0^{\pi} e^{jz\cos\theta\cos\psi}\, J_{\nu-\frac{1}{2}}(z\sin\theta\sin\psi)\, C_n^{\nu}(\cos\theta)\sin^{\nu+\frac{1}{2}}\theta\, d\theta$$

$$= \sqrt{\frac{2\pi}{z}}\, j^n \cdot \sin^{\nu-\frac{1}{2}}\psi \cdot C_n^{\nu}(\cos\psi)\, J_{\nu+n}(z)$$

For $\nu = 1/2$.

$$\int_0^{\pi} e^{jz\cos\theta\cos\psi}\, J_0(z\sin\theta\sin\psi)\, C_n^{1/2}(\cos\theta)\sin\theta\, d\theta = \sqrt{\frac{2\pi}{z}}\, j^n C_n^{1/2}(\cos\psi)\, J_{\frac{1}{2}+n}(z)$$

From page 50 of reference 12.

$$\frac{1}{(1-2at+a^2)^{\nu}} = \sum_n a^n C_n^{\nu}(t), \quad \text{where} \begin{cases} C_n^{1/2}(t) = P_n^0(t) \\ C_0^{1/2}(t) = P_0^0(t) = 1. \end{cases}$$

Thus for $n=0$, the above integral reduces to

$$\int_0^{\pi} e^{jz\cos\theta\cos\psi}\, J_0(z\sin\theta\sin\psi)\, C_0^{1/2}(\cos\theta)\sin\theta\, d\theta = \sqrt{\frac{2\pi}{z}}\, j^0 C_0^{1/2}(\cos\psi)\, J_{\frac{1}{2}}(z)$$

or $$\int_0^{\pi} e^{jz\cos\theta\cos\psi} J_0(z\sin\theta\sin\psi)\sin\theta\, d\theta = \sqrt{\frac{2\pi}{z}}\, J_{\frac{1}{2}}(z)$$

Since $j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x) \Rightarrow \sqrt{\frac{2\pi}{z}}\, J_{\frac{1}{2}}(x) = 2 j_0(z)$

then $$\int_0^{\pi} e^{jz\cos\theta\cos\psi}\, J_0(z\sin\theta\sin\psi)\sin\theta\, d\theta = 2\, j_0(z) = 2\,\frac{\sin(z)}{z}$$

Making a change of variable of the form $\theta' = \theta - \pi/2$.

the above integral can be written as

$$\int_{-\pi/2}^{\pi/2} e^{jz\sin\theta'\cos\psi}\, J_0(z\cos\theta'\sin\psi)\cos\theta' d\theta' = 2\,\frac{\sin(z)}{z}$$

Going through another change of variable of the form

$\xi = -\sin\theta' \Rightarrow d\xi = -\cos\theta' d\theta'$, $\cos\theta' = \sqrt{1-\sin^2(\theta')}$

the integral reduces to

$$\int_{-1}^{+1} e^{jz\xi\cos\psi} \cdot J_0(z\sin\psi\sqrt{1-\xi^2})\, d\xi = 2\,\frac{\sin(z)}{z}$$

By letting $\xi = \frac{2}{L} z' \Rightarrow d\xi = \frac{2}{L} dz'$, the integral can be written as

$$\int_{-L/2}^{L/2} e^{jz(\frac{2}{L}z')\cos\psi} \cdot J_0\left[z\sin\psi\cdot\sqrt{1-(\tfrac{2}{L}z')^2}\right]\frac{2}{L}\, dz' = 2\,\frac{\sin(z)}{z}$$

(Continued)

7-17. (cont'd) Lastly by letting

$$\left.\begin{aligned} z\sin\psi &= j\pi B \\ \tfrac{2}{L}(z\cos\psi) &= k\cos\theta \end{aligned}\right\} \Rightarrow z^2(\sin^2\psi+\cos^2\psi) = z^2 = \left(\tfrac{\pi l}{\lambda}\cos\theta\right)^2 - (\pi B)^2$$

or

$$z = \sqrt{\left(\tfrac{\pi l}{\lambda}\cos\theta\right)^2 - (\pi B)^2} = \sqrt{\mu^2 - (\pi B)^2}$$

Therefore

$$\int_{-l/2}^{l/2} e^{jkz'\cos\theta}\, J_0\left(j\pi B\sqrt{1-(\tfrac{2}{L}z')^2}\right) dz' = l\,\frac{\sin\left[\sqrt{\mu^2-(\pi B)^2}\right]}{\sqrt{\mu^2-(\pi B)^2}}$$

where $\mu = \pi\frac{l}{\lambda}\cos\theta$

7-18. For a -30 dB side lobe level

$$R_0 = 10^{30/20} = 10^{1.5} = 31.62$$

and $31.62 = 4.603\,\dfrac{\sinh(\pi B)}{\pi B} \Rightarrow \pi B = 4.009 \Rightarrow B = 1.2761$

The excitation coefficients are obtained using (7-32).

For a spacing of $d = \lambda/2$ between the elements

$$2a_1 = I_n(z')\big|_{z'=0} = J_0(j4.009) = 11.40 \Rightarrow a_1 = 5.70$$

$$a_2 = I_n(z')\big|_{z'=\pm\lambda/2} = J_0(j3.882) = 10.2$$

$$a_3 = I_n(z')\big|_{z'=\pm\lambda} = J_0(j3.472) = 7.2$$

$$a_4 = I_n(z')\big|_{z'=\pm\frac{3\lambda}{2}} = J_0(j2.652) = 3.7$$

$$a_5 = I_n(z')\big|_{z'=\pm 2\lambda} = J_0(j0) = 1$$

The array factor is given by (6-61b) and (6-61c), or

$$AF(\theta) = a_1 + a_2\cos(2\mu) + a_3\cos(4\mu) + a_4\cos(6\mu) + a_5\cos(8\mu)$$

where $\mu = \frac{\pi d}{\lambda}\cos\theta = \frac{\pi}{2}\cos\theta$

The patterns computed for the continuous distribution and the discretized array of $N = 9$ elements with $d = \lambda/2$, are shown plotted in Fig. P7.18. The agreement between the two is not as good as that in Fig. 7.10.

(Continued)

7-18. (Cont'd)

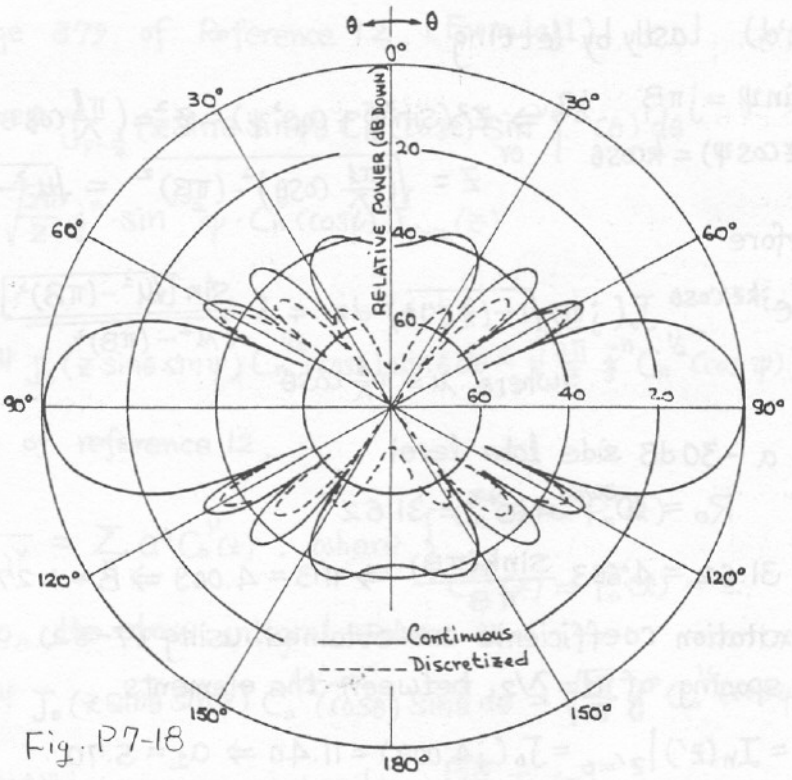



Fig. P7-18

7-19. For a -40dB side lobe level

$R_0 = 100$ and $B = 1.7415$

The excitation coefficients are equal to

$$2a_1 = J_0(j5.471) = 41.55 \Rightarrow a_1 = 20.775 = 20.775\angle 0^\circ$$

$$a_2 = J_0(j4.738) = 21.58 \Rightarrow 21.58\angle 0^\circ$$

$$a_3 = J_0(j0) = 1 = 1\angle 0^\circ$$

The array factor is given by

$$AF(\theta,\phi) = a_1 + a_2\cos(2u) + a_3\cos(4u)$$

where $u = \frac{\pi}{4}\sin\theta\cos\phi$

7-20. Using (7-1) for $\phi_n(z') = 0$

$$SF(\theta) = \int_{-\ell/2}^{\ell/2} I_n(z')\, e^{jkz'\cos\theta}\, dz'$$

Uniform; $I_n(z') = I_0$

$$SF(\theta) = \int_{-\ell/2}^{\ell/2} I_0\, e^{jkz'\cos\theta}\, dz' = I_0 \int_{-\ell/2}^{\ell/2} \frac{e^{jkz'\cos\theta}}{jk\cos\theta}\, d(jk\cos\theta\, z')$$

$$= I_0 \ell \left[\frac{\sin(\frac{k\ell}{2}\cos\theta)}{\frac{k\ell}{2}\cos\theta}\right] = I_0 \ell \left[\frac{\sin(u)}{u}\right] \quad \text{where } u = \pi\left(\frac{\ell}{\lambda}\right)\cos\theta$$

In a similar manner, it can be shown that for

Triangular; $I_n(z') = I_1\left(1 - \frac{2}{\ell}|z'|\right)$

$$SF(\theta) = I_1\left(\frac{\ell}{2}\right)\frac{[\sin(\frac{u}{2})]}{[\frac{u}{2}]}, \quad \text{where } u = \pi\left(\frac{\ell}{\lambda}\right)\cos\theta$$

Cosine; $I_n(z') = I_2 \cos(\pi z'/\ell)$

$$SF(\theta) = I_2 \cdot \ell \cdot \left(\frac{\pi}{2}\right) \cdot \frac{\cos u}{(\pi/2)^2 - u^2}, \quad \text{where } u = \pi\left(\frac{\ell}{\lambda}\right)\cos\theta$$

Cosine Squared; $I_n(z') = I_3 \cos^2\left(\pi \frac{z'}{\ell}\right)$

$$SF(\theta) = I_3\left(\frac{\ell}{2}\right)\left[\frac{\sin u}{u}\right]\left[\frac{\pi^2}{\pi^2 - u^2}\right]$$

7-21. For $\ell = 4\lambda$ and Table 7.1

a. Uniform distribution:
HPBW = 50.6/4 = 12.65°
FNBW = 114.6/4 = 28.65°
FSLMM = -13.2 dB
$D_0$ = 2(4) = 8 = 9.03 dB

b. Triangular distribution:
HPBW = 73.4/4 = 18.35°
FNBW = 229.2/4 = 57.30°
FSLMM = -26.4 dB
$D_0$ = 0.75(8) = 6.80 = 8.33 dB

c. Cosine distribution;
HPBW = 68.8/4 = 17.20°
FNBW = 171.9/4 = 42.98°
FSLMM = -23.2 dB
$D_0$ = 0.810(8) = 6.48 = 8.12 dB

d. Cosine-Squared distribution;
HPBW = 83.2/4 = 20.80°
FNBW = 229.2/4 = 57.30°
FSLMM = -31.5 dB
$D_0$ = 0.667(8) = 5.34 = 7.27 dB

7-22. Using (7-45), the space factor for each distribution can be derived as follows;

Uniform: $A_n(\rho') = I_0$

$$SF(\theta) = 2\pi I_0 \int_0^a \rho' J_0(k\rho'\sin\theta)\, d\rho'$$

Using Equation ( V-22 ) from Appendix V, or

$$\int_0^a x^{p+1} J_p(\alpha x)\, dx \Big|_{p=0} = \int_0^a x J_0(\alpha x)\, dx = \frac{1}{\alpha} x J_1(\alpha x)\Big|_0^a$$

the space factor reduces to

$$SF(\theta) = 2\pi I_0 \left\{ \frac{a}{k\sin\theta} \left[ J_1(ka\sin\theta) - \overset{0}{J_1(0)} \right] \right\} = 2\pi a^2 I_0 \frac{J_1(ka\sin\theta)}{ka\sin\theta}$$

$$= 2\pi a^2 I_0 \frac{J_1(u)}{u} \quad , \text{ where } u = ka\sin\theta = \frac{2\pi a}{\lambda}\sin\theta$$

Radial ; $A_n(\rho') = I_1 [1 - (\rho'/a)^2]$

$$SF(\theta) = 2\pi I_1 \int_0^a [1 - (\rho'/a)^2]\, J_0(k\rho'\sin\theta)\, \rho'\, d\rho'$$

Using the integral

$$g_p(u = ka\sin\theta) = 2\pi a^2 \int_0^1 (1 - r^2)^p J_0(ur)\, r\, dr$$

$$= 2\pi \int_0^a \left[1 - \left(\frac{\rho'}{a}\right)^2\right]^p J_0(k\rho'\sin\theta)\, \rho'\, d\rho' = 2\pi a^2 \frac{2^p\, p!\, J_{p+1}(u)}{u^{p+1}}$$

the space factor reduces to

$$SF(\theta) = 4\pi a^2 I_1 \left[ \frac{J_2(u)}{u} \right], \quad \text{where } u = \left(\frac{2\pi a}{\lambda}\right)\sin\theta = ka\sin\theta$$

Radial Squared;

$$A_n(\rho') = I_2 [1 - (\rho'/a)^2]^2$$

Using the same procedure as for the radial, it can be shown that

$$SF(\theta) = 16\pi a^2 I_2 \left[ \frac{J_3(u)}{u} \right], \quad u = \left(\frac{2\pi a}{\lambda}\right)\sin\theta = ka\sin\theta$$

7-23. Using the equations from Table 7.2 with $a = 2\lambda$, the desired parameters for each distribution are as follows:

a. Uniform;

HPBW $= 29.2/2 = 14.60°$

FNBW $= 69.9/2 = 34.95°$

FSLMM $= -17.6$ dB

$D_o = (4\pi)^2 = 157.91 = 21.98$ dB

b. Radial taper;

HPBW $= 36.4/2 = 18.20°$

FNBW $= 93.4/2 = 46.70°$

FSLMM $= -24.6$ dB

$D_o = 0.75\,(4\pi)^2 = 118.44 = 20.73$ dB

c. Radial taper - Squared;

HPBW $= 42.1/2 = 21.05°$

FNBW $= 116.3/2 = 58.15°$

FSLMM $= -30.6$ dB

$D_o = 0.56\,(4\pi)^2 = 88.43 = 19.47$ dB

8-1. $\frac{1}{4\pi}\int_{-l/2}^{l/2} I(z')\left[\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)\frac{e^{jkR}}{R}\right]dz' = -j\omega\epsilon E_z^i(\rho=a),\quad R^2=(z-z')^2+a^2$

$$\frac{\partial}{\partial z}\left(\frac{e^{jkR}}{R}\right) = -(1+j\beta R)\frac{(z-z')}{R^3}e^{j\beta R}$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{e^{j\beta R}}{R}\right) = -(1+j\beta R)\frac{e^{j\beta R}}{R^3} - \left[j\beta\frac{(z-z')^2}{R^4} - (1+j\beta R)(3+j\beta R)\frac{(z-z')^2}{R^5}\right]e^{-j\beta R}$$

$$= -(1+j\beta R)\frac{e^{j\beta R}}{R^3} - \left[j\beta(R^2-a^2)R - (3+j4\beta R-(\beta R)^2)(R^2-a^2)\right]\frac{e^{j\beta R}}{R^5}$$

$$= -(1+j\beta R)\frac{e^{-j\beta R}}{R^3} - (R^2-a^2)\left[(\beta R)^2 - 3(1+j\beta R)\right]\frac{e^{j\beta R}}{R^5}$$

$$= \frac{e^{-j\beta R}}{R^5}\left[-R^2(1+j\beta R) - (R^2-a^2)(\beta R)^2 + 3(R^2-a^2)(1+j\beta R)\right]$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{e^{-j\beta R}}{R}\right) = \frac{e^{-j\beta R}}{R^5}\left\{(1+j\beta R)(2R^2-3a^2) - (R^2-a^2)(\beta R)^2\right\}$$

Therefore

$$\frac{1}{4\pi}\int_{-l/2}^{l/2} I(z')\left[\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)\frac{e^{-j\beta R}}{R}\right]dz' = \frac{1}{4\pi}\int_{-l/2}^{l/2} I(z')\frac{e^{-j\beta R}}{R^5}\left\{(1+j\beta R)(2R^2-3a^2) - k^2(kR)^2\right.$$

$$\left. + a^2(\beta R)^2 + \beta^2R^4\right\}dz' = \frac{1}{4\pi}\int_{-l/2}^{l/2} I(z')\frac{e^{-j\beta R}}{R^5}\left[(1+j\beta R)(2R^2-3a^2) + (\beta aR)^2\right]dz'$$

8-2 According to (8-20) we have that

$$\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)\int_{-l/2}^{l/2} I_z(z')\frac{e^{-j\beta R}}{4\pi R}dz' = -j\omega\epsilon E_z^i(\rho=a) = -j\frac{\beta}{\eta}E_z^i \qquad (1)$$

The above equation is an inhomogeneous ordinary harmonic differential equation whose solution is obtained as the sum of complementary and particular solutions. Thus we can write the solution as

$$\int_{-l/2}^{l/2} I_z(z')\frac{e^{-j\beta R}}{4\pi R}dz' = j\frac{\beta}{\eta}\underbrace{\left[B\cos(\beta z)+C\sin(\beta z)\right]}_{\text{complementary}} - j\frac{1}{\eta}\underbrace{\int_0^z E_z^i\sin\left[\beta(z-z')\right]dz'}_{\text{particular solution}} \qquad (2)$$

At $z=0$, the particular solution vanishes. Therefore (2) then reduces to

$$\int_{-l/2}^{l/2} I_z(z')\frac{e^{-j\beta R}}{4\pi R}dz' = -j\frac{\beta}{\eta}\left[B\cos(\beta z)+C\sin(\beta z)\right] = -j\frac{1}{\eta}\left[\beta B\cos(\beta z)+\beta C\sin(\beta z)\right]$$

$$= -j\sqrt{\frac{\epsilon}{\mu}}\left[B_1\cos(\beta z)+C_1\sin(\beta z)\right]$$

8-3. $\underline{E}_f = \hat{a}_\rho \frac{V_i}{2\rho' \ln(b/a)}$, $\underline{M}_f = -2\hat{n} \times \underline{E}_f = -2\hat{a}_z \times \hat{a}_\rho E_\rho = -\hat{a}_\phi \frac{V_i}{\rho' \ln(b/a)}$, (1)

$$M_\phi = -\frac{V_i}{\rho' \ln(b/a)}$$

The solution presented here is that reported in the following journal Paper:

L. L. Tsai," A numerical solution for the near and far fields of an annular ring of magnetic current", IEEE Transaction Antennas and Propagation, Vol. AP-20, No.5, Sept. 1972, PP. 569-576.

The radiated electric and magnetic fields can be found using

$$\underline{E} = -j\omega \underline{A} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla \cdot \underline{A}) - \frac{1}{\epsilon}\nabla \times \underline{F} \quad (2a)$$

$$\underline{H} = -j\omega \underline{F} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla \cdot \underline{F}) + \frac{1}{\mu}\nabla \times \underline{A} \quad (2b)$$

where $\underline{F} = \frac{\epsilon}{4\pi}\iint_S \underline{M}_f \frac{e^{jkR}}{R} ds'$ (2c)

$$\underline{A} = \frac{\mu}{4\pi}\iint_S \underline{J} \frac{e^{jkR}}{R} ds' \quad (2d)$$

Since $\underline{J}=0$, then $\underline{A}=0$ and (2a) and (2b) reduce to

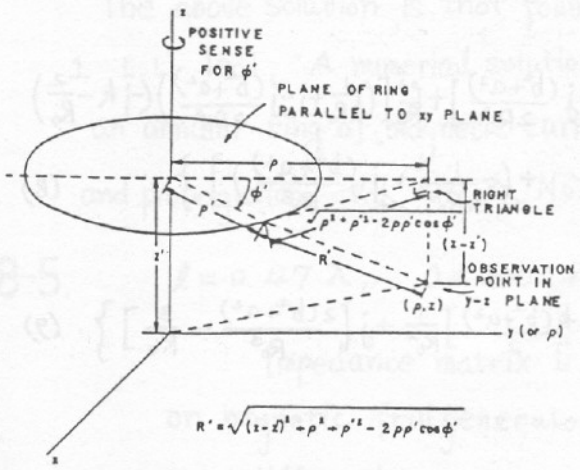
$$\underline{E} = -\frac{1}{\epsilon}\nabla \times \underline{F} \quad (3a)$$

$$\underline{H} = -j\omega \underline{F} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla \cdot \underline{F}) \quad (3b)$$

To evaluate (2c) the differential area $ds'$ on the frill is

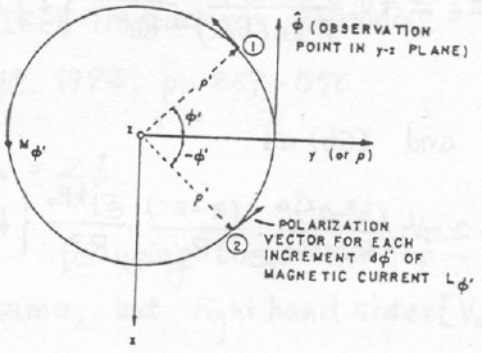
$$ds' = \rho' d\rho' d\phi' \quad (4)$$

Dividing the frill into $d\rho'$ wide magnetic ring current, the contribution of each ring will be evaluated first. This is accomplished by integrating with respect to $\phi'$ before summing over all the rings.




Geometry for Ring

Fig. P8.3




Polarization for source and observation

(Continued)

8-3 (Cont'd) The problem possesses symmetry since there are no $\phi$ variations. Therefore the observation can be located at any plane. Without loss of generality we choose the yz plane for the observations. For this plane $\rho = y$ and the $\underline{F}$ potential is $\phi$-polarized ($F_\phi$). Using (1) and (2c) we can then write that

$$F_\phi = -\frac{\epsilon}{2\pi}\frac{1}{\ln(b/a)}\int_0^b\int_0^\pi \cos\phi' \frac{e^{-jkR'}}{R'}\,d\phi'\,d\rho' \tag{5}$$

where $$R' = [(z-z')^2 + \rho^2 - (\rho')^2 - 2\rho\rho'\cos\phi']^{1/2} \tag{5a}$$

Because the ring is on the xy-plane, each pair of incremental current element $d\phi'$ (symmetrically located at $\pm\phi'$) has only a net $\hat{a}_\phi$ contribution. This leads to the $2\cos\phi'$ factor with an integration in $\phi'$ from 0 to $\pi$.

Now using (2a) and (2b) in cylindrical coordinates, we can write that

$$E_z = -\frac{1}{\epsilon}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\phi) \tag{6a}$$

$$E_\rho = \frac{1}{\epsilon}\frac{\partial}{\partial z}(F_\phi) \tag{6b}$$

$$H_\phi = -j\omega F_\phi \tag{6c}$$

$$E_\phi = H_\rho = H_z = 0 \tag{6d}$$

Closed form expressions for the electric and magnetic fields can be derived when $\rho \gg b$ without making far field approximations. From (5a)

$$R_0 = [(z-z')^2 + \rho^2]^{1/2}$$

After suitable approximations on (5) we can write that

$$F_\phi(\rho, z) \cong -\frac{\epsilon}{4\ln(b/a)}\,k\rho\,\frac{e^{-jkR_0}}{R_0^2}(b^2-a^2)\left\{-j\frac{(b^2+a^2)}{4R_0^2} + \frac{1}{2kR_0} + j\frac{1}{2}\right\} \tag{7}$$

Using (7), we can write (6a) as

$$E_z \simeq +V_i\frac{k(b^2-a^2)}{8\ln(b/a)}\,\frac{e^{-jkR_0}}{R_0^2}\left\{2\left[\frac{1}{kR_0}+j-j\frac{(b^2+a^2)}{2R_0^2}\right]+\frac{\rho^2}{R_0}\left[\left(\frac{1}{kR_0}+j-j\frac{(b^2+a^2)}{2R_0^2}\right)\left(-jk-\frac{2}{R_0}\right)\right.\right.$$
$$\left.\left.+\left(-\frac{1}{kR_0^2}+j\frac{(b^2+a^2)}{R_0^3}\right)\right]\right\} \tag{8}$$

and (6b) as

$$E_\rho \simeq -V_i\frac{(b^2-a^2)\rho}{8\ln(b/a)}\cdot\frac{(z-z')}{R_0}\cdot\frac{e^{-jkR_0}}{R_0^2}\left\{k-\left[\frac{3}{k}+\frac{k(b^2+a^2)}{2}\right]\frac{1}{R_0^2}+j\left[\frac{2(b^2+a^2)}{R_0^3}-\frac{3}{R_0}\right]\right\} \tag{9}$$

8-4. On the axis of the ring ($\rho=0$), the symmetry of the problem along the $\phi$ direction requires that $E_\rho(\rho=0, z)=0$. However equations (8) and (9) of the solution of Problem 8.3 are no longer accurate when $\rho \to 0$. Therefore we will derive a simpler form for the field which will be accurate only for $\rho=0$. when $\rho=0$, we have that

$$R' = [(z-z')^2+(\rho')^2]^{1/2} \tag{1}$$

Therefore

$$F_\phi(0,z) = -\frac{\epsilon}{4\pi}\frac{1}{\ln(b/a)}\int_a^b \frac{e^{-jkR'}}{R'}\left[\int_0^{2\pi}\cos\phi' d\phi'\right]d\rho' = 0 \tag{2}$$

Then

$$H_\phi(0,z) = -j\omega F_\phi = 0 \tag{3}$$

From (6a) of the solution of Problem 8.3

$$E_z(0,z) = -\frac{1}{\epsilon}\frac{F_\phi}{\rho} - \frac{1}{\epsilon}\frac{\partial F_\phi}{\partial \rho} = -\frac{2}{\epsilon}\frac{\partial F_\phi}{\partial \rho}\Big|_{\rho=0} \tag{4}$$

In deriving (4), L'Hospital's rule was used to take care of the $\frac{0}{0}$ term. Using (2) and (4) and interchanging the order of integration and differentiation, it can be shown that (4) can be written as

$$E_z(0,z) = \frac{1}{\pi}\frac{1}{\ln(b/a)}\int_0^b\int_0^\pi \cos\phi'\left[\frac{\partial}{\partial\rho}\frac{e^{-jkR'}}{R'}\right]_{\rho=0} d\phi' d\rho'$$

$$E_z(0,z) = \frac{1}{2\ln(b/a)}\left\{\frac{e^{-jk[(z-z')^2+a^2]^{1/2}}}{[(z-z')^2+a^2]^{1/2}} - \frac{e^{-jk[(z-z')^2+b^2]^{1/2}}}{[(z-z')^2+b^2]^{1/2}}\right\} \tag{5}$$

The above solution is that found in:

1 L.L. Tsai, "A numerical solution for the near and far fields of an annular ring of magnetic current", IEEE Transactions Antennas and propagation, Vol. AP-20, No. 5, Sept. 1972. pp. 569-576

8-5. $\ell = 0.47\lambda$, $a = 0.005\lambda$, $N = 21$.

impedance matrix $[Z_{mn}]$ for using the delta-gap or magnetic frill generator is same, but right hand sides $[V_m]$ are different,

(Continued)

```
Z( 1, 1)= -0.9604352E+03 -0.2944177E+00j  Z( 4, 2)=  0.4475594E+02 -0.2920930E+00j
Z( 1, 2)=  0.4184994E+03 -0.2938329E+00j  Z( 4, 3)=  0.4184994E+03 -0.2938329E+00j
Z( 1, 3)=  0.4475594E+02 -0.2920930E+00j  Z( 4, 4)= -0.9604352E+03 -0.2944177E+00j
Z( 1, 4)=  0.1323194E+02 -0.2892089E+00j  Z( 4, 5)=  0.4184994E+03 -0.2938329E+00j
Z( 1, 5)=  0.5799302E+01 -0.2852054E+00j  Z( 4, 6)=  0.4475594E+02 -0.2920930E+00j
Z( 1, 6)=  0.3131011E+01 -0.2801163E+00j  Z( 4, 7)=  0.1323194E+02 -0.2892089E+00j
Z( 1, 7)=  0.1919739E+01 -0.2739840E+00j  Z( 4, 8)=  0.5799302E+01 -0.2852054E+00j
Z( 1, 8)=  0.1278995E+01 -0.2668600E+00j  Z( 4, 9)=  0.3131011E+01 -0.2801163E+00j
Z( 1, 9)=  0.9010428E+00 -0.2588035E+00j  Z( 4,10)=  0.1919739E+01 -0.2739840E+00j
Z( 1,10)=  0.6590829E+00 -0.2498812E+00j  Z( 4,11)=  0.1278995E+01 -0.2668600E+00j
Z( 1,11)=  0.4940229E+00 -0.2401658E+00j  Z( 4,12)=  0.9010428E+00 -0.2588035E+00j
Z( 1,12)=  0.3756340E+00 -0.2297367E+00j  Z( 4,13)=  0.6590829E+00 -0.2498812E+00j
Z( 1,13)=  0.2872772E+00 -0.2186776E+00j  Z( 4,14)=  0.4940229E+00 -0.2401658E+00j
Z( 1,14)=  0.2192344E+00 -0.2070767E+00j  Z( 4,15)=  0.3756340E+00 -0.2297367E+00j
Z( 1,15)=  0.1655427E+00 -0.1950254E+00j  Z( 4,16)=  0.2872772E+00 -0.2186776E+00j
Z( 1,16)=  0.1223871E+00 -0.1826168E+00j  Z( 4,17)=  0.2192344E+00 -0.2070767E+00j
Z( 1,17)=  0.8723569E-01 -0.1699458E+00j  Z( 4,18)=  0.1655427E+00 -0.1950254E+00j
Z( 1,18)=  0.5835235E-01 -0.1571072E+00j  Z( 4,19)=  0.1223871E+00 -0.1826168E+00j
Z( 1,19)=  0.3451012E-01 -0.1441954E+00j  Z( 4,20)=  0.8723569E-01 -0.1699458E+00j
Z( 1,20)=  0.1481668E-01 -0.1313030E+00j  Z( 4,21)=  0.5835235E-01 -0.1571072E+00j
Z( 1,21)= -0.1395117E-02 -0.1185201E+00j  Z( 5, 1)=  0.5799302E+01 -0.2852054E+00j
Z( 2, 1)=  0.4184994E+03 -0.2938329E+00j  Z( 5, 2)=  0.1323194E+02 -0.2892089E+00j
Z( 2, 2)= -0.9604352E+03 -0.2944177E+00j  Z( 5, 3)=  0.4475594E+02 -0.2920930E+00j
Z( 2, 3)=  0.4184994E+03 -0.2938329E+00j  Z( 5, 4)=  0.4184994E+03 -0.2938329E+00j
Z( 2, 4)=  0.4475594E+02 -0.2920930E+00j  Z( 5, 5)= -0.9604352E+03 -0.2944177E+00j
Z( 2, 5)=  0.1323194E+02 -0.2892089E+00j  Z( 5, 6)=  0.4184994E+03 -0.2938329E+00j
Z( 2, 6)=  0.5799302E+01 -0.2852054E+00j  Z( 5, 7)=  0.4475594E+02 -0.2920930E+00j
Z( 2, 7)=  0.3131011E+01 -0.2801163E+00j  Z( 5, 8)=  0.1323194E+02 -0.2892089E+00j
Z( 2, 8)=  0.1919739E+01 -0.2739840E+00j  Z( 5, 9)=  0.5799302E+01 -0.2852054E+00j
Z( 2, 9)=  0.1278995E+01 -0.2668600E+00j  Z( 5,10)=  0.3131011E+01 -0.2801163E+00j
Z( 2,10)=  0.9010428E+00 -0.2588035E+00j  Z( 5,11)=  0.1919739E+01 -0.2739840E+00j
Z( 2,11)=  0.6590829E+00 -0.2498812E+00j  Z( 5,12)=  0.1278995E+01 -0.2668600E+00j
Z( 2,12)=  0.4940229E+00 -0.2401658E+00j  Z( 5,13)=  0.9010428E+00 -0.2588035E+00j
Z( 2,13)=  0.3756340E+00 -0.2297367E+00j  Z( 5,14)=  0.6590829E+00 -0.2498812E+00j
Z( 2,14)=  0.2872772E+00 -0.2186776E+00j  Z( 5,15)=  0.4940229E+00 -0.2401658E+00j
Z( 2,15)=  0.2192344E+00 -0.2070767E+00j  Z( 5,16)=  0.3756340E+00 -0.2297367E+00j
Z( 2,16)=  0.1655427E+00 -0.1950254E+00j  Z( 5,17)=  0.2872772E+00 -0.2186776E+00j
Z( 2,17)=  0.1223871E+00 -0.1826168E+00j  Z( 5,18)=  0.2192344E+00 -0.2070767E+00j
Z( 2,18)=  0.8723569E-01 -0.1699458E+00j  Z( 5,19)=  0.1655427E+00 -0.1950254E+00j
Z( 2,19)=  0.5835235E-01 -0.1571072E+00j  Z( 5,20)=  0.1223871E+00 -0.1826168E+00j
Z( 2,20)=  0.3451012E-01 -0.1441954E+00j  Z( 5,21)=  0.8723569E-01 -0.1699458E+00j
Z( 2,21)=  0.1481668E-01 -0.1313030E+00j  Z( 6, 1)=  0.3131011E+01 -0.2801163E+00j
Z( 3, 1)=  0.4475594E+02 -0.2920930E+00j  Z( 6, 2)=  0.5799302E+01 -0.2852054E+00j
Z( 3, 2)=  0.4184994E+03 -0.2938329E+00j  Z( 6, 3)=  0.1323194E+02 -0.2892089E+00j
Z( 3, 3)= -0.9604352E+03 -0.2944177E+00j  Z( 6, 4)=  0.4475594E+02 -0.2920930E+00j
Z( 3, 4)=  0.4184994E+03 -0.2938329E+00j  Z( 6, 5)=  0.4184994E+03 -0.2938329E+00j
Z( 3, 5)=  0.4475594E+02 -0.2920930E+00j  Z( 6, 6)= -0.9604352E+03 -0.2944177E+00j
Z( 3, 6)=  0.1323194E+02 -0.2892089E+00j  Z( 6, 7)=  0.4184994E+03 -0.2938329E+00j
Z( 3, 7)=  0.5799302E+01 -0.2852054E+00j  Z( 6, 8)=  0.4475594E+02 -0.2920930E+00j
Z( 3, 8)=  0.3131011E+01 -0.2801163E+00j  Z( 6, 9)=  0.1323194E+02 -0.2892089E+00j
Z( 3, 9)=  0.1919739E+01 -0.2739840E+00j  Z( 6,10)=  0.5799302E+01 -0.2852054E+00j
Z( 3,10)=  0.1278995E+01 -0.2668600E+00j  Z( 6,11)=  0.3131011E+01 -0.2801163E+00j
Z( 3,11)=  0.9010428E+00 -0.2588035E+00j  Z( 6,12)=  0.1919739E+01 -0.2739840E+00j
Z( 3,12)=  0.6590829E+00 -0.2498812E+00j  Z( 6,13)=  0.1278995E+01 -0.2668600E+00j
Z( 3,13)=  0.4940229E+00 -0.2401658E+00j  Z( 6,14)=  0.9010428E+00 -0.2588035E+00j
Z( 3,14)=  0.3756340E+00 -0.2297367E+00j  Z( 6,15)=  0.6590829E+00 -0.2498812E+00j
Z( 3,15)=  0.2872772E+00 -0.2186776E+00j  Z( 6,16)=  0.4940229E+00 -0.2401658E+00j
Z( 3,16)=  0.2192344E+00 -0.2070767E+00j  Z( 6,17)=  0.3756340E+00 -0.2297367E+00j
Z( 3,17)=  0.1655427E+00 -0.1950254E+00j  Z( 6,18)=  0.2872772E+00 -0.2186776E+00j
Z( 3,18)=  0.1223871E+00 -0.1826168E+00j  Z( 6,19)=  0.2192344E+00 -0.2070767E+00j
Z( 3,19)=  0.8723569E-01 -0.1699458E+00j  Z( 6,20)=  0.1655427E+00 -0.1950254E+00j
Z( 3,20)=  0.5835235E-01 -0.1571072E+00j  Z( 6,21)=  0.1223871E+00 -0.1826168E+00j
Z( 3,21)=  0.3451012E-01 -0.1441954E+00j  Z( 7, 1)=  0.1919739E+01 -0.2739840E+00j
Z( 4, 1)=  0.1323194E+02 -0.2892089E+00j  Z( 7, 2)=  0.3131011E+01 -0.2801163E+00j
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Z( 7, 3)=  0.5799302E+01 -0.2852054E+00j  Z(10, 4)=  0.1919739E+01 -0.2739840E+00j
Z( 7, 4)=  0.1323194E+02 -0.2892089E+00j  Z(10, 5)=  0.3131011E+01 -0.2801163E+00j
Z( 7, 5)=  0.4475594E+02 -0.2920930E+00j  Z(10, 6)=  0.5799302E+01 -0.2852054E+00j
Z( 7, 6)=  0.4184994E+03 -0.2938329E+00j  Z(10, 7)=  0.1323194E+02 -0.2892089E+00j
Z( 7, 7)= -0.9604352E+03 -0.2944177E+00j  Z(10, 8)=  0.4475594E+02 -0.2920930E+00j
Z( 7, 8)=  0.4184994E+03 -0.2938329E+00j  Z(10, 9)=  0.4184994E+03 -0.2938329E+00j
Z( 7, 9)=  0.4475594E+02 -0.2920930E+00j  Z(10,10)= -0.9604352E+03 -0.2944177E+00j
Z( 7,10)=  0.1323194E+02 -0.2892089E+00j  Z(10,11)=  0.4184994E+03 -0.2938329E+00j
Z( 7,11)=  0.5799302E+01 -0.2852054E+00j  Z(10,12)=  0.4475594E+02 -0.2920930E+00j
Z( 7,12)=  0.3131011E+01 -0.2801163E+00j  Z(10,13)=  0.1323194E+02 -0.2892089E+00j
Z( 7,13)=  0.1919739E+01 -0.2739840E+00j  Z(10,14)=  0.5799302E+01 -0.2852054E+00j
Z( 7,14)=  0.1278995E+01 -0.2668600E+00j  Z(10,15)=  0.3131011E+01 -0.2801163E+00j
Z( 7,15)=  0.9010428E+00 -0.2588035E+00j  Z(10,16)=  0.1919739E+01 -0.2739840E+00j
Z( 7,16)=  0.6590829E+00 -0.2498812E+00j  Z(10,17)=  0.1278995E+01 -0.2668600E+00j
Z( 7,17)=  0.4940229E+00 -0.2401658E+00j  Z(10,18)=  0.9010428E+00 -0.2588035E+00j
Z( 7,18)=  0.3756340E+00 -0.2297367E+00j  Z(10,19)=  0.6590829E+00 -0.2498812E+00j
Z( 7,19)=  0.2872772E+00 -0.2186776E+00j  Z(10,20)=  0.4940229E+00 -0.2401658E+00j
Z( 7,20)=  0.2192344E+00 -0.2070767E+00j  Z(10,21)=  0.3756340E+00 -0.2297367E+00j
Z( 7,21)=  0.1655427E+00 -0.1950254E+00j  Z(11, 1)=  0.4940229E+00 -0.2401658E+00j
Z( 8, 1)=  0.1278995E+01 -0.2668600E+00j  Z(11, 2)=  0.6590829E+00 -0.2498812E+00j
Z( 8, 2)=  0.1919739E+01 -0.2739840E+00j  Z(11, 3)=  0.9010428E+00 -0.2588035E+00j
Z( 8, 3)=  0.3131011E+01 -0.2801163E+00j  Z(11, 4)=  0.1278995E+01 -0.2668600E+00j
Z( 8, 4)=  0.5799302E+01 -0.2852054E+00j  Z(11, 5)=  0.1919739E+01 -0.2739840E+00j
Z( 8, 5)=  0.1323194E+02 -0.2892089E+00j  Z(11, 6)=  0.3131011E+01 -0.2801163E+00j
Z( 8, 6)=  0.4475594E+02 -0.2920930E+00j  Z(11, 7)=  0.5799302E+01 -0.2852054E+00j
Z( 8, 7)=  0.4184994E+03 -0.2938329E+00j  Z(11, 8)=  0.1323194E+02 -0.2892089E+00j
Z( 8, 8)= -0.9604352E+03 -0.2944177E+00j  Z(11, 9)=  0.4475594E+02 -0.2920930E+00j
Z( 8, 9)=  0.4184994E+03 -0.2938329E+00j  Z(11,10)=  0.4184994E+03 -0.2938329E+00j
Z( 8,10)=  0.4475594E+02 -0.2920930E+00j  Z(11,11)= -0.9604352E+03 -0.2944177E+00j
Z( 8,11)=  0.1323194E+02 -0.2892089E+00j  Z(11,12)=  0.4184994E+03 -0.2938329E+00j
Z( 8,12)=  0.5799302E+01 -0.2852054E+00j  Z(11,13)=  0.4475594E+02 -0.2920930E+00j
Z( 8,13)=  0.3131011E+01 -0.2801163E+00j  Z(11,14)=  0.1323194E+02 -0.2892089E+00j
Z( 8,14)=  0.1919739E+01 -0.2739840E+00j  Z(11,15)=  0.5799302E+01 -0.2852054E+00j
Z( 8,15)=  0.1278995E+01 -0.2668600E+00j  Z(11,16)=  0.3131011E+01 -0.2801163E+00j
Z( 8,16)=  0.9010428E+00 -0.2588035E+00j  Z(11,17)=  0.1919739E+01 -0.2739840E+00j
Z( 8,17)=  0.6590829E+00 -0.2498812E+00j  Z(11,18)=  0.1278995E+01 -0.2668600E+00j
Z( 8,18)=  0.4940229E+00 -0.2401658E+00j  Z(11,19)=  0.9010428E+00 -0.2588035E+00j
Z( 8,19)=  0.3756340E+00 -0.2297367E+00j  Z(11,20)=  0.6590829E+00 -0.2498812E+00j
Z( 8,20)=  0.2872772E+00 -0.2186776E+00j  Z(11,21)=  0.4940229E+00 -0.2401658E+00j
Z( 8,21)=  0.2192344E+00 -0.2070767E+00j  Z(12, 1)=  0.3756340E+00 -0.2297367E+00j
Z( 9, 1)=  0.9010428E+00 -0.2588035E+00j  Z(12, 2)=  0.4940229E+00 -0.2401658E+00j
Z( 9, 2)=  0.1278995E+01 -0.2668600E+00j  Z(12, 3)=  0.6590829E+00 -0.2498812E+00j
Z( 9, 3)=  0.1919739E+01 -0.2739840E+00j  Z(12, 4)=  0.9010428E+00 -0.2588035E+00j
Z( 9, 4)=  0.3131011E+01 -0.2801163E+00j  Z(12, 5)=  0.1278995E+01 -0.2668600E+00j
Z( 9, 5)=  0.5799302E+01 -0.2852054E+00j  Z(12, 6)=  0.1919739E+01 -0.2739840E+00j
Z( 9, 6)=  0.1323194E+02 -0.2892089E+00j  Z(12, 7)=  0.3131011E+01 -0.2801163E+00j
Z( 9, 7)=  0.4475594E+02 -0.2920930E+00j  Z(12, 8)=  0.5799302E+01 -0.2852054E+00j
Z( 9, 8)=  0.4184994E+03 -0.2938329E+00j  Z(12, 9)=  0.1323194E+02 -0.2892089E+00j
Z( 9, 9)= -0.9604352E+03 -0.2944177E+00j  Z(12,10)=  0.4475594E+02 -0.2920930E+00j
Z( 9,10)=  0.4184994E+03 -0.2938329E+00j  Z(12,11)=  0.4184994E+03 -0.2938329E+00j
Z( 9,11)=  0.4475594E+02 -0.2920930E+00j  Z(12,12)= -0.9604352E+03 -0.2944177E+00j
Z( 9,12)=  0.1323194E+02 -0.2892089E+00j  Z(12,13)=  0.4184994E+03 -0.2938329E+00j
Z( 9,13)=  0.5799302E+01 -0.2852054E+00j  Z(12,14)=  0.4475594E+02 -0.2920930E+00j
Z( 9,14)=  0.3131011E+01 -0.2801163E+00j  Z(12,15)=  0.1323194E+02 -0.2892089E+00j
Z( 9,15)=  0.1919739E+01 -0.2739840E+00j  Z(12,16)=  0.5799302E+01 -0.2852054E+00j
Z( 9,16)=  0.1278995E+01 -0.2668600E+00j  Z(12,17)=  0.3131011E+01 -0.2801163E+00j
Z( 9,17)=  0.9010428E+00 -0.2588035E+00j  Z(12,18)=  0.1919739E+01 -0.2739840E+00j
Z( 9,18)=  0.6590829E+00 -0.2498812E+00j  Z(12,19)=  0.1278995E+01 -0.2668600E+00j
Z( 9,19)=  0.4940229E+00 -0.2401658E+00j  Z(12,20)=  0.9010428E+00 -0.2588035E+00j
Z( 9,20)=  0.3756340E+00 -0.2297367E+00j  Z(12,21)=  0.6590829E+00 -0.2498812E+00j
Z( 9,21)=  0.2872772E+00 -0.2186776E+00j  Z(13, 1)=  0.2872772E+00 -0.2186776E+00j
Z(10, 1)=  0.6590829E+00 -0.2498812E+00j  Z(13, 2)=  0.3756340E+00 -0.2297367E+00j
Z(10, 2)=  0.9010428E+00 -0.2588035E+00j  Z(13, 3)=  0.4940229E+00 -0.2401658E+00j
Z(10, 3)=  0.1278995E+01 -0.2668600E+00j  Z(13, 4)=  0.6590829E+00 -0.2498812E+00j
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Z(13, 5)=  0.9010428E+00 -0.2588035E+00j Z(16, 6)=  0.4940229E+00 -0.2401658E+00j
Z(13, 6)=  0.1278995E+01 -0.2668600E+00j Z(16, 7)=  0.6590829E+00 -0.2498812E+00j
Z(13, 7)=  0.1919739E+01 -0.2739840E+00j Z(16, 8)=  0.9010428E+00 -0.2588035E+00j
Z(13, 8)=  0.3131011E+01 -0.2801163E+00j Z(16, 9)=  0.1278995E+01 -0.2668600E+00j
Z(13, 9)=  0.5799302E+01 -0.2852054E+00j Z(16,10)=  0.1919739E+01 -0.2739840E+00j
Z(13,10)=  0.1323194E+02 -0.2892089E+00j Z(16,11)=  0.3131011E+01 -0.2801163E+00j
Z(13,11)=  0.4475594E+02 -0.2920930E+00j Z(16,12)=  0.5799302E+01 -0.2852054E+00j
Z(13,12)=  0.4184994E+03 -0.2938329E+00j Z(16,13)=  0.1323194E+02 -0.2892089E+00j
Z(13,13)= -0.9604352E+03 -0.2944177E+00j Z(16,14)=  0.4475594E+02 -0.2920930E+00j
Z(13,14)=  0.4184994E+03 -0.2938329E+00j Z(16,15)=  0.4184994E+03 -0.2938329E+00j
Z(13,15)=  0.4475594E+02 -0.2920930E+00j Z(16,16)= -0.9604352E+03 -0.2944177E+00j
Z(13,16)=  0.1323194E+02 -0.2892089E+00j Z(16,17)=  0.4184994E+03 -0.2938329E+00j
Z(13,17)=  0.5799302E+01 -0.2852054E+00j Z(16,18)=  0.4475594E+02 -0.2920930E+00j
Z(13,18)=  0.3131011E+01 -0.2801163E+00j Z(16,19)=  0.1323194E+02 -0.2892089E+00j
Z(13,19)=  0.1919739E+01 -0.2739840E+00j Z(16,20)=  0.5799302E+01 -0.2852054E+00j
Z(13,20)=  0.1278995E+01 -0.2668600E+00j Z(16,21)=  0.3131011E+01 -0.2801163E+00j
Z(13,21)=  0.9010428E+00 -0.2588035E+00j Z(17, 1)=  0.8723569E-01 -0.1699458E+00j
Z(14, 1)=  0.2192344E+00 -0.2070767E+00j Z(17, 2)=  0.1223871E+00 -0.1826168E+00j
Z(14, 2)=  0.2872772E+00 -0.2186776E+00j Z(17, 3)=  0.1655427E+00 -0.1950254E+00j
Z(14, 3)=  0.3756340E+00 -0.2297367E+00j Z(17, 4)=  0.2192344E+00 -0.2070767E+00j
Z(14, 4)=  0.4940229E+00 -0.2401658E+00j Z(17, 5)=  0.2872772E+00 -0.2186776E+00j
Z(14, 5)=  0.6590829E+00 -0.2498812E+00j Z(17, 6)=  0.3756340E+00 -0.2297367E+00j
Z(14, 6)=  0.9010428E+00 -0.2588035E+00j Z(17, 7)=  0.4940229E+00 -0.2401658E+00j
Z(14, 7)=  0.1278995E+01 -0.2668600E+00j Z(17, 8)=  0.6590829E+00 -0.2498812E+00j
Z(14, 8)=  0.1919739E+01 -0.2739840E+00j Z(17, 9)=  0.9010428E+00 -0.2588035E+00j
Z(14, 9)=  0.3131011E+01 -0.2801163E+00j Z(17,10)=  0.1278995E+01 -0.2668600E+00j
Z(14,10)=  0.5799302E+01 -0.2852054E+00j Z(17,11)=  0.1919739E+01 -0.2739840E+00j
Z(14,11)=  0.1323194E+02 -0.2892089E+00j Z(17,12)=  0.3131011E+01 -0.2801163E+00j
Z(14,12)=  0.4475594E+02 -0.2920930E+00j Z(17,13)=  0.5799302E+01 -0.2852054E+00j
Z(14,13)=  0.4184994E+03 -0.2938329E+00j Z(17,14)=  0.1323194E+02 -0.2892089E+00j
Z(14,14)= -0.9604352E+03 -0.2944177E+00j Z(17,15)=  0.4475594E+02 -0.2920930E+00j
Z(14,15)=  0.4184994E+03 -0.2938329E+00j Z(17,16)=  0.4184994E+03 -0.2938329E+00j
Z(14,16)=  0.4475594E+02 -0.2920930E+00j Z(17,17)= -0.9604352E+03 -0.2944177E+00j
Z(14,17)=  0.1323194E+02 -0.2892089E+00j Z(17,18)=  0.4184994E+03 -0.2938329E+00j
Z(14,18)=  0.5799302E+01 -0.2852054E+00j Z(17,19)=  0.4475594E+02 -0.2920930E+00j
Z(14,19)=  0.3131011E+01 -0.2801163E+00j Z(17,20)=  0.1323194E+02 -0.2892089E+00j
Z(14,20)=  0.1919739E+01 -0.2739840E+00j Z(17,21)=  0.5799302E+01 -0.2852054E+00j
Z(14,21)=  0.1278995E+01 -0.2668600E+00j Z(18, 1)=  0.5835235E-01 -0.1571072E+00j
Z(15, 1)=  0.1655427E+00 -0.1950254E+00j Z(18, 2)=  0.8723569E-01 -0.1699458E+00j
Z(15, 2)=  0.2192344E+00 -0.2070767E+00j Z(18, 3)=  0.1223871E+00 -0.1826168E+00j
Z(15, 3)=  0.2872772E+00 -0.2186776E+00j Z(18, 4)=  0.1655427E+00 -0.1950254E+00j
Z(15, 4)=  0.3756340E+00 -0.2297367E+00j Z(18, 5)=  0.2192344E+00 -0.2070767E+00j
Z(15, 5)=  0.4940229E+00 -0.2401658E+00j Z(18, 6)=  0.2872772E+00 -0.2186776E+00j
Z(15, 6)=  0.6590829E+00 -0.2498812E+00j Z(18, 7)=  0.3756340E+00 -0.2297367E+00j
Z(15, 7)=  0.9010428E+00 -0.2588035E+00j Z(18, 8)=  0.4940229E+00 -0.2401658E+00j
Z(15, 8)=  0.1278995E+01 -0.2668600E+00j Z(18, 9)=  0.6590829E+00 -0.2498812E+00j
Z(15, 9)=  0.1919739E+01 -0.2739840E+00j Z(18,10)=  0.9010428E+00 -0.2588035E+00j
Z(15,10)=  0.3131011E+01 -0.2801163E+00j Z(18,11)=  0.1278995E+01 -0.2668600E+00j
Z(15,11)=  0.5799302E+01 -0.2852054E+00j Z(18,12)=  0.1919739E+01 -0.2739840E+00j
Z(15,12)=  0.1323194E+02 -0.2892089E+00j Z(18,13)=  0.3131011E+01 -0.2801163E+00j
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Z(15,14)=  0.4184994E+03 -0.2938329E+00j Z(18,15)=  0.1323194E+02 -0.2892089E+00j
Z(15,15)= -0.9604352E+03 -0.2944177E+00j Z(18,16)=  0.4475594E+02 -0.2920930E+00j
Z(15,16)=  0.4184994E+03 -0.2938329E+00j Z(18,17)=  0.4184994E+03 -0.2938329E+00j
Z(15,17)=  0.4475594E+02 -0.2920930E+00j Z(18,18)= -0.9604352E+03 -0.2944177E+00j
Z(15,18)=  0.1323194E+02 -0.2892089E+00j Z(18,19)=  0.4184994E+03 -0.2938329E+00j
Z(15,19)=  0.5799302E+01 -0.2852054E+00j Z(18,20)=  0.4475594E+02 -0.2920930E+00j
Z(15,20)=  0.3131011E+01 -0.2801163E+00j Z(18,21)=  0.1323194E+02 -0.2892089E+00j
Z(15,21)=  0.1919739E+01 -0.2739840E+00j Z(19, 1)=  0.3451012E-01 -0.1441954E+00j
Z(16, 1)=  0.1223871E+00 -0.1826168E+00j Z(19, 2)=  0.5835235E-01 -0.1571072E+00j
Z(16, 2)=  0.1655427E+00 -0.1950254E+00j Z(19, 3)=  0.8723569E-01 -0.1699458E+00j
Z(16, 3)=  0.2192344E+00 -0.2070767E+00j Z(19, 4)=  0.1223871E+00 -0.1826168E+00j
Z(16, 4)=  0.2872772E+00 -0.2186776E+00j Z(19, 5)=  0.1655427E+00 -0.1950254E+00j
Z(16, 5)=  0.3756340E+00 -0.2297367E+00j Z(19, 6)=  0.2192344E+00 -0.2070767E+00j
```

```
Z(19, 7)=  0.2872772E+00 -0.2186776E+00j
Z(19, 8)=  0.3756340E+00 -0.2297367E+00j
Z(19, 9)=  0.4940229E+00 -0.2401658E+00j
Z(19,10)=  0.6590829E+00 -0.2498812E+00j
Z(19,11)=  0.9010428E+00 -0.2588035E+00j
Z(19,12)=  0.1278995E+01 -0.2668600E+00j
Z(19,13)=  0.1919739E+01 -0.2739840E+00j
Z(19,14)=  0.3131011E+01 -0.2801163E+00j
Z(19,15)=  0.5799302E+01 -0.2852054E+00j
Z(19,16)=  0.1323194E+02 -0.2892089E+00j
Z(19,17)=  0.4475594E+02 -0.2920930E+00j
Z(19,18)=  0.4184994E+03 -0.2938329E+00j
Z(19,19)= -0.9604352E+03 -0.2944177E+00j
Z(19,20)=  0.4184994E+03 -0.2938329E+00j
Z(19,21)=  0.4475594E+02 -0.2920930E+00j
Z(20, 1)=  0.1481668E-01 -0.1313030E+00j
Z(20, 2)=  0.3451012E-01 -0.1441954E+00j
Z(20, 3)=  0.5835235E-01 -0.1571072E+00j
Z(20, 4)=  0.8723569E-01 -0.1699458E+00j
Z(20, 5)=  0.1223871E+00 -0.1826168E+00j
Z(20, 6)=  0.1655427E+00 -0.1950254E+00j
Z(20, 7)=  0.2192344E+00 -0.2070767E+00j
Z(20, 8)=  0.2872772E+00 -0.2186776E+00j
Z(20, 9)=  0.3756340E+00 -0.2297367E+00j
Z(20,10)=  0.4940229E+00 -0.2401658E+00j
Z(20,11)=  0.6590829E+00 -0.2498812E+00j
Z(20,12)=  0.9010428E+00 -0.2588035E+00j
Z(20,13)=  0.1278995E+01 -0.2668600E+00j
Z(20,14)=  0.1919739E+01 -0.2739840E+00j
Z(20,15)=  0.3131011E+01 -0.2801163E+00j
Z(20,16)=  0.5799302E+01 -0.2852054E+00j
Z(20,17)=  0.1323194E+02 -0.2892089E+00j
Z(20,18)=  0.4475594E+02 -0.2920930E+00j
Z(20,19)=  0.4184994E+03 -0.2938329E+00j
Z(20,20)= -0.9604352E+03 -0.2944177E+00j
Z(20,21)=  0.4184994E+03 -0.2938329E+00j
Z(21, 1)= -0.1395117E-02 -0.1185201E+00j
Z(21, 2)=  0.1481668E-01 -0.1313030E+00j
Z(21, 3)=  0.3451012E-01 -0.1441954E+00j
Z(21, 4)=  0.5835235E-01 -0.1571072E+00j
Z(21, 5)=  0.8723569E-01 -0.1699458E+00j
Z(21, 6)=  0.1223871E+00 -0.1826168E+00j
Z(21, 7)=  0.1655427E+00 -0.1950254E+00j
Z(21, 8)=  0.2192344E+00 -0.2070767E+00j
Z(21, 9)=  0.2872772E+00 -0.2186776E+00j
Z(21,10)=  0.3756340E+00 -0.2297367E+00j
Z(21,11)=  0.4940229E+00 -0.2401658E+00j
Z(21,12)=  0.6590829E+00 -0.2498812E+00j
Z(21,13)=  0.9010428E+00 -0.2588035E+00j
Z(21,14)=  0.1278995E+01 -0.2668600E+00j
Z(21,15)=  0.1919739E+01 -0.2739840E+00j
Z(21,16)=  0.3131011E+01 -0.2801163E+00j
Z(21,17)=  0.5799302E+01 -0.2852054E+00j
Z(21,18)=  0.1323194E+02 -0.2892089E+00j
Z(21,19)=  0.4475594E+02 -0.2920930E+00j
Z(21,20)=  0.4184994E+03 -0.2938329E+00j
Z(21,21)= -0.9604352E+03 -0.2944177E+00j
```

8-6. $\ell = \lambda/50$, $a = 0.005\lambda$, $N = 21$.

Delta Gap

$Z_{in} = 8.17 \times 10^{-6} - j\,2.75 \times 10^{-2}$ ohms

Magnetic Frill

$Z_{in} = 0.2199 - j\,421.4$ ohms

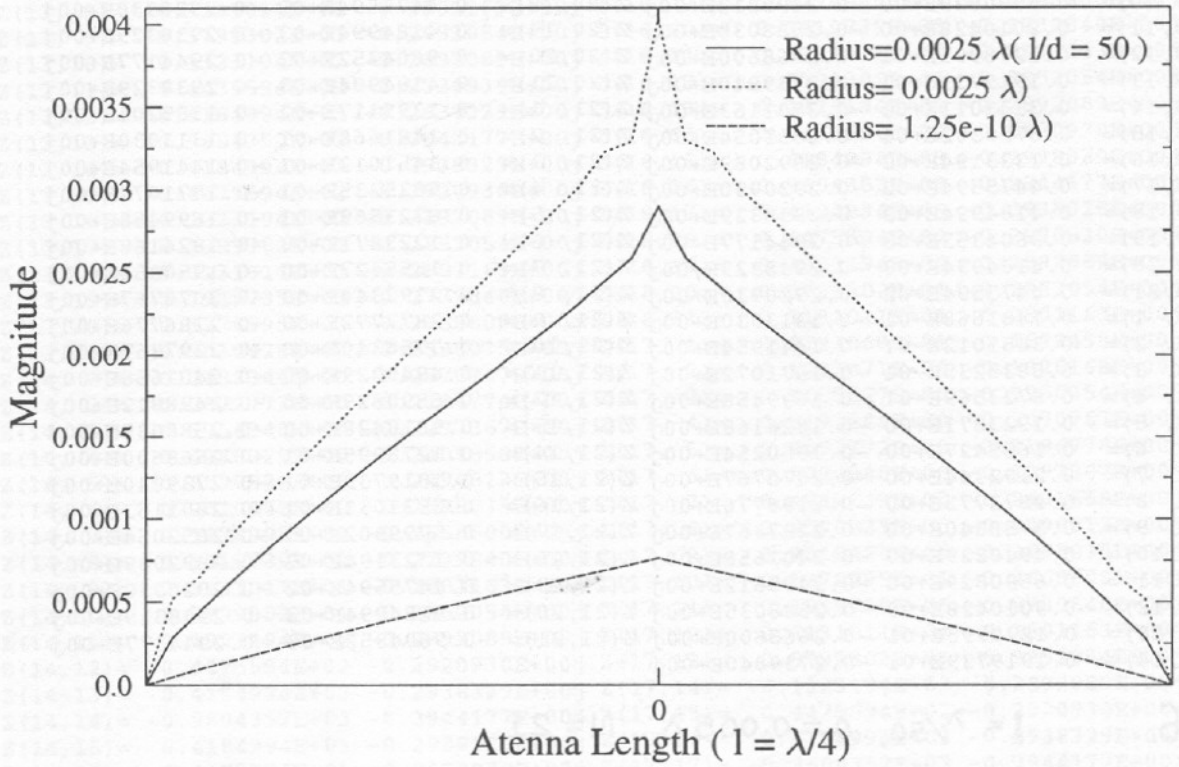
8-7.

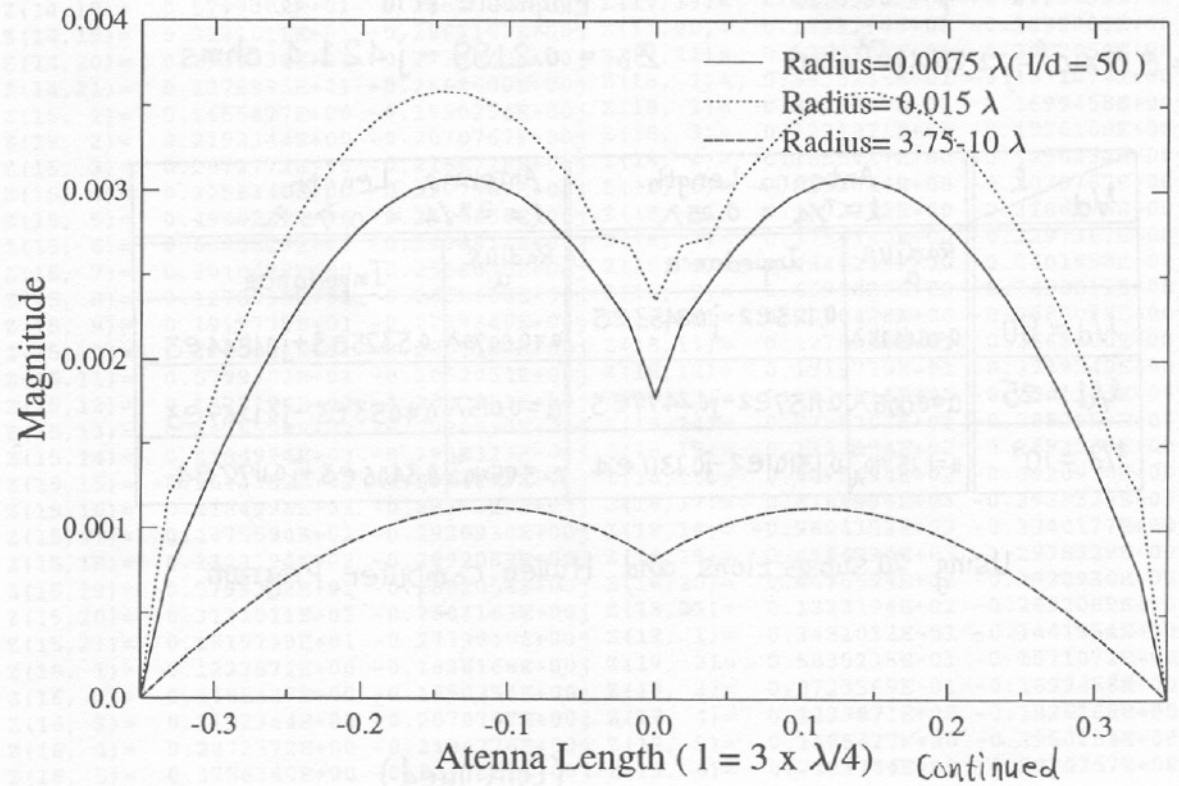
| $\ell$ / $\ell/d$ | Antenna Length $\ell = \lambda/4 = 0.25\lambda$ | | Antenna Length $\ell = 3\lambda/4 = 0.75\lambda$ | |
|---|---|---|---|---|
| | Radius a | Impedance | Radius a | Impedance |
| $\ell/d = 50$ | $a = 0.0025\lambda$ | 0.123e2 - j0.3453e3 | $a = 0.0075\lambda$ | 0.5325e3 + j0.1844e3 |
| $\ell/d = 25$ | $a = 0.005\lambda$ | 0.1137e2 - j0.2477e3 | $a = 0.015\lambda$ | 0.4053e3 - j0.1269e3 |
| $\ell/d = 10^9$ | $a = 1.25 \times 10^{-10}\lambda$ | 0.13101e2 - j0.1311e4 | $a = 3.75 \times 10^{-10}\lambda$ | 0.3406e3 + j0.1177e4 |

Using 20 subsections and Hallen computer Program

(Continued)

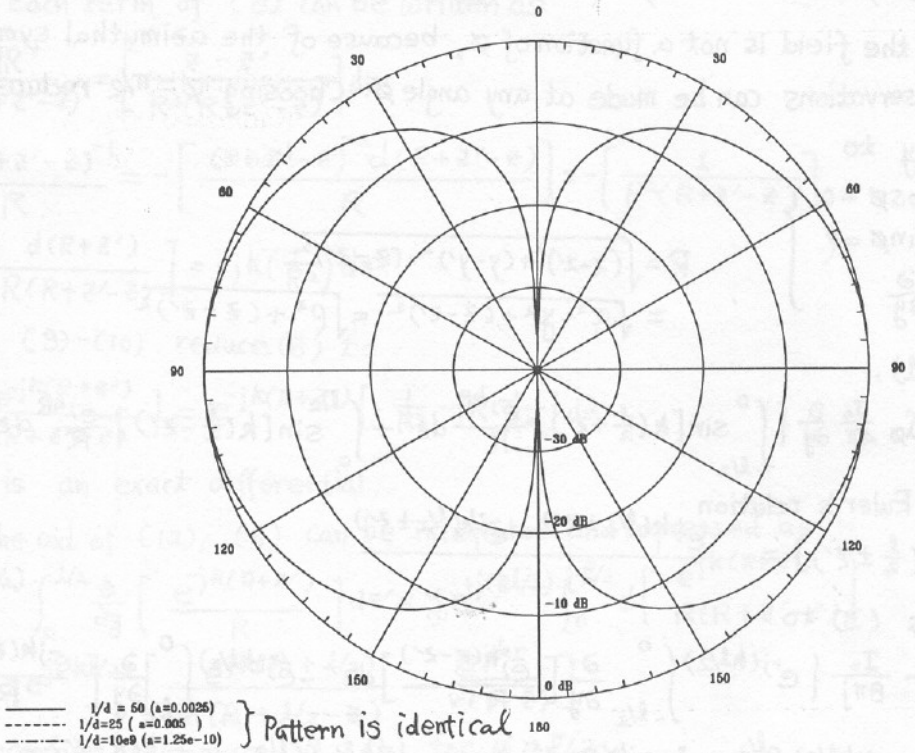
8-7 (Cont'd)



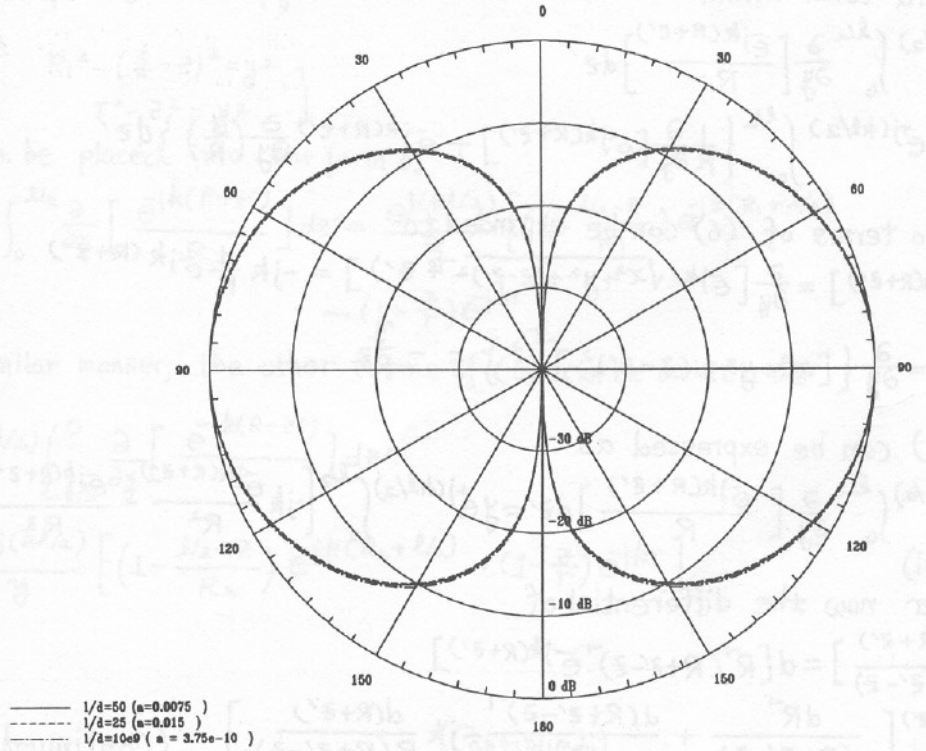




Continued

8-7(Cont'd) Radiation Pattern Of Dipole Antenna ( L = $\lambda/4$ ) (Prob. 8-7)




Radiation Pattern Of Dipole Antenna ( L =3 x $\lambda/4$ ) (Prob. 8-7)



8-8 $$\underline{H} = \frac{1}{\mu}\nabla \times \underline{A} = -\hat{a}_\phi \frac{1}{\mu}\frac{\partial A_z}{\partial \rho} \quad (1)$$

Since the field is not a function of $\phi$, because of the azimuthal symmetry, the observations can be made at any angle $\phi$. Choosing $\phi = \pi/2$ reduces $x$ and $y$ to

$$\left.\begin{aligned} x &= \rho\cos\phi = 0 \\ y &= \rho\sin\phi = \rho \\ \frac{\partial}{\partial\rho} &= \frac{\partial}{\partial y} \end{aligned}\right\}, \quad \begin{aligned} R &= \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \\ &= \sqrt{x^2+y^2+(z-z')^2} = \sqrt{\rho^2+(z-z')^2} \end{aligned} \quad (2)$$

and (1),

$$\underline{H} = -\hat{a}_\phi \frac{I_0}{4\pi}\frac{\partial}{\partial y}\left\{\int_{-l/2}^{0} \sin\left[k\left(\frac{l}{2}+z'\right)\right]\frac{e^{-jkR}}{R}dz' + \int_{0}^{l/2}\sin\left[k\left(\frac{l}{2}-z'\right)\right]\frac{e^{-jkR}}{R}dz'\right\} \quad (3)$$

Using Euler's relation

$$\sin\left[k\left(\frac{l}{2}\pm z'\right)\right] = \frac{e^{jk\left(l/2\pm z'\right)} - e^{-jk\left(l/2\pm z'\right)}}{2j} \quad (4)$$

reduces (3) to

$$\begin{aligned} H_\phi = -\frac{I_0}{8\pi j}\Bigg\{ & e^{+j(kl/2)}\int_{-l/2}^{0}\frac{\partial}{\partial y}\left[\frac{e^{-jk(R-z')}}{R}\right]dz' - e^{-j(kl/2)}\int_{-l/2}^{0}\frac{\partial}{\partial y}\left[\frac{e^{-jk(R+z')}}{R}\right]dz' \\ & + e^{+j(kl/2)}\int_{0}^{l/2}\frac{\partial}{\partial y}\left[\frac{e^{-jk(R+z')}}{R}\right]dz' - e^{-j(kl/2)}\int_{0}^{l/2}\frac{\partial}{\partial y}\left[\frac{e^{-jk(R-z')}}{R}\right]dz'\Bigg\} \end{aligned} \quad (5)$$

The third term within the brackets can be written as

$$\begin{aligned} & e^{+j(kl/2)}\int_{0}^{l/2}\frac{\partial}{\partial y}\left[\frac{e^{-jk(R+z')}}{R}\right]dz' \\ & = e^{+j(kl/2)}\int_{0}^{l/2}\left\{\frac{1}{R}\frac{\partial}{\partial y}\left[e^{-jk(R+z')}\right] + e^{-jk(R+z')}\frac{\partial}{\partial y}\left(\frac{1}{R}\right)\right\}dz' \end{aligned} \quad (6)$$

The two terms of (6) can be expanded to

$$\frac{\partial}{\partial y}\left[e^{-jk(R+z')}\right] = \frac{\partial}{\partial y}\left[e^{-jk\left(\sqrt{x^2+y^2+(z-z')^2}+z'\right)}\right] = -jk\frac{y}{R}e^{-jk(R+z')} \quad (6\text{-a})$$

$$\frac{\partial}{\partial y}\left(\frac{1}{R}\right) = \frac{\partial}{\partial y}\left\{\left[x^2+y^2+(z-z')^2\right]^{-1/2}\right\} = -\frac{y}{R^3} \quad (6\text{b})$$

Thus (6) can be expressed as

$$e^{+j(kl/2)}\int_{0}^{l/2}\frac{\partial}{\partial y}\left[\frac{e^{-jk(R+z')}}{R}\right]dz' = y e^{+j(kl/2)}\int_{0}^{l/2}\left[-jk\frac{e^{-jk(R+z')}}{R^2} - \frac{e^{-jk(R+z')}}{R^3}\right]dz' \quad (7)$$

Consider now the differential of

$$\begin{aligned} d\left[\frac{e^{-jk(R+z')}}{R(R+z'-z)}\right] &= d\left[R^{-1}(R+z'-z)^{-1}e^{-jk(R+z')}\right] \\ &= e^{-jk(R+z')}\left[\frac{dR^{-1}}{(R+z'-z)} + \frac{d(R+z'-z)^{-1}}{R} - jk\frac{d(R+z')}{R(R+z'-z)}\right] \end{aligned} \quad (8)$$

(Continued)

8-8 (Cont'd)

Each term of (8) can be written as

$$\frac{dR^{-1}}{(R+z'-z)} = \left[\frac{z-z'}{R^3(R+z'-z)}\right]dz' \tag{9}$$

$$\frac{d(R+z'-z)^{-1}}{R} = -\left[\frac{(R+z'-z)^{-2}d(R+z'-z)}{R}\right] = -\left[\frac{1}{R^2(R+z'-z)}\right] \tag{10}$$

$$-jk\left[\frac{d(R+z')}{R(R+z'-z)}\right] = -jk\left(\frac{1}{R^2}\right)dz' \tag{11}$$

Adding (9)-(10) reduce (8) to

$$d\left[\frac{e^{-jk(R+z')}}{R(R+z'-z)}\right] = e^{-jk(R+z')}\left[-\frac{1}{R^3} - jk\frac{1}{R^2}\right]dz' \tag{12}$$

which is an exact differential.

With the aid of (12), (6) can be integrated and expressed as

$$e^{+j(kl/2)}\int_0^{l/2}\frac{\partial}{\partial y}\left[\frac{e^{-jk(R+z')}}{R}\right]dz' = y\,e^{+j(kl/2)}\int_0^{l/2} d\left[\frac{e^{-jk(R+z')}}{R(R+z'-z)}\right]$$

$$= y\,e^{j(kl/2)}\left[\frac{e^{-jk(R_1+l/2)}}{R_1(R_1+l/2-z)} - \frac{e^{-jkr}}{r(r-z)}\right] \tag{13}$$

where according to Figure 8.1 (b) for $\phi = \pi/2$

$$R_1 = \sqrt{x^2+y^2+(z-\tfrac{l}{2})^2} = \sqrt{y^2+(z-\tfrac{l}{2})^2} \tag{14}$$

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{y^2+z^2} \tag{15}$$

Because $R_1^2-(\frac{l}{2}-z)^2 = y^2$ and $r^2-z^2 = y^2$

(13) can be placed into the form of

$$e^{+j(kl/2)}\int_0^{l/2}\frac{\partial}{\partial y}\left[\frac{e^{-jk(R+z')}}{R}\right]dz' = \frac{e^{j(kl/2)}}{y}\left[\left(1-\frac{l/2-z}{R_1}\right)e^{-jk(R_1+l/2)} - \left(1+\frac{z}{r}\right)e^{-jkr}\right] \tag{16}$$

In a similar manner, the other terms of (5) can be written as

$$+e^{+j(kl/2)}\int_{-l/2}^{0}\frac{\partial}{\partial y}\left[\frac{e^{-jk(R-z')}}{R}\right]dz'$$

$$= \frac{e^{j(kl/2)}}{y}\left[\left(1-\frac{l/2+z}{R_2}\right)e^{-jk(R_2+l/2)} - \left(1-\frac{z}{r}\right)e^{-jkr}\right] \tag{16a}$$

(Continued)

8-8 (Cont'd)

$$-e^{-j(kl/2)}\int_{-l/2}^{0}\frac{\partial}{\partial y}\left[\frac{e^{-jk(R+z')}}{R}\right]dz'$$

$$=\frac{e^{-j(kl/2)}}{y}\left[\left(1+\frac{l/2+z}{R_3}\right)e^{-jk(R_2-l/2)}-\left(1+\frac{z}{r}\right)e^{-jkr}\right] \tag{16b}$$

$$-e^{-j(kl/2)}\int_{0}^{l/2}\frac{\partial}{\partial y}\left[\frac{e^{-jk(R-z')}}{R}\right]dz'$$

$$=\frac{e^{-j(kl/2)}}{y}\left[\left(1+\frac{l/2-z}{R_1}\right)e^{-jk(R_1-l/2)}-\left(1-\frac{z}{r}\right)e^{-jkr}\right] \tag{16c}$$

where $$R_2=\sqrt{x^2+y^2+(z+\tfrac{l}{2})^2}=\sqrt{\rho^2+(z+\tfrac{l}{2})^2} \tag{16d}$$

Using (16)-(16c) reduces (5) to

$$H_\phi=-\frac{I_0}{4\pi j}\frac{1}{y}\left[e^{-jkR_1}+e^{-jkR_2}-2\cos\left(\tfrac{kl}{2}\right)e^{-jkr}\right] \tag{17}$$

$$\underline{E}=\frac{1}{j\omega\epsilon}\nabla\times\underline{H}=\frac{1}{j\omega\epsilon}\left\{\hat{a}_\rho\left[\frac{1}{\rho}\cancelto{0}{\frac{\partial H_z}{\partial\phi}}-\frac{\partial H_\phi}{\partial z}\right]+\hat{a}_\phi\left[\cancelto{0}{\frac{\partial H_\rho}{\partial z}}-\cancelto{0}{\frac{\partial H_z}{\partial\rho}}\right]+\hat{a}_z\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho H_\phi)-\cancelto{0}{\frac{\partial H_\rho}{\partial\phi}}\right]\right\} \tag{18}$$

which reduces in the yz plane ($x=0$) to

$$E_\rho=E_y=\frac{1}{j\omega\epsilon}\frac{\partial H_\phi}{\partial z} \tag{19}$$

$$E_z=\frac{1}{j\omega\epsilon}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\phi)=\frac{1}{j\omega\epsilon}\frac{1}{y}\frac{\partial}{\partial y}(yH_\phi) \tag{20}$$

since $\rho=y$. Thus

$$E_\rho=\frac{1}{j\omega\epsilon}\frac{\partial}{\partial z}\left\{-\frac{I_0}{4\pi j}\frac{1}{y}\left[e^{-jkR_1}+e^{-jkR_2}-2\cos\left(\tfrac{kl}{2}\right)e^{-jkr}\right]\right\}$$

$$=j\frac{\eta I_0}{4\pi y}\left[(z-\tfrac{l}{2})\frac{e^{-jkR_1}}{R_1}+(z+l)\frac{e^{-jkR_2}}{R_2}-2z\cos\left(\tfrac{kl}{2}\right)\frac{e^{-jkr}}{r}\right] \tag{21}$$

$$E_z=\frac{1}{j\omega\epsilon}\frac{1}{y}\frac{\partial}{\partial y}\left\{-\frac{I_0}{4\pi j}\left[\frac{e^{-jkR_1}}{R_1}+\frac{e^{-jkR_2}}{R_2}-2\cos\left(\tfrac{kl}{2}\right)e^{-jkr}\right]\right\}$$

$$=-j\frac{\eta I_0}{4\pi}\left[\frac{e^{-jkR_1}}{R_1}+\frac{e^{-jkR_2}}{R_2}-2\cos\left(\tfrac{kl}{2}\right)\frac{e^{-jkr}}{r}\right] \tag{22}$$

8-9. $R_{im} = R_r = 60\{C + \ln(kl) - C_i(kl) + \frac{1}{2}\sin(kl)[S_i(2kl) - 2S_i(kl)]$

$+ \frac{1}{2}\cos(kl)[C + \ln(kl/2) + C_i(2kl) - 2C_i(kl)]\}$

$X_{im} = 30\{2S_i(kl) + \cos(kl)[2S_i(kl) - S_i(2kl)]$

$- \sin(kl)[2C_i(kl) - C_i(2kl) - C_i(\frac{2ka^2}{l})]\}$

where $C = 0.5772$

$C_i(\frac{2ka^2}{l}) = 0.577 + \ln(\frac{2ka^2}{l}) - C_{in}(\frac{2ka^2}{l})$ according to (4-69)

$R_{in} = R_{im}/(\sin^2(kl/2))$, $X_{in} = X_{im}/(\sin^2(kl/2))$

a. $l = \lambda/4$: $kl = \pi/2$, $2kl = \pi$, $2ka^2/l = \pi/8 = 0.12566$, $kl/2 = \pi/4$

$C_i(kl) = 0.47$ $\quad \ln(kl/2) = -0.24156$

$C_i(2kl) = 0.0774$ $\quad \ln(kl) = 0.45158$

$S_i(kl) = 1.37$ $\quad C_i(2ka^2/l) \simeq 0.577 + \ln(0.12566) \simeq 0.577 - 2.0741$

$S_i(2kl) = 1.851$ $\quad = -1.517$

Substituting these numbers into the previous equations leads to

$R_{im} \simeq 6.844$, $\quad R_{in} = 2(6.844) = 13.688$

$X_{im} \simeq 10.710$, $\quad X_{in} = 2(10.710) = 21.420$

b. $l = \lambda/2$: $kl = \pi$, $2kl = 2\pi$, $2ka^2/l = \pi/16 = 0.06283$, $kl/2 = \pi/8$

$C_i(kl) = 0.07367$ $\quad \ln(kl/2) = 0.45158$

$C_i(2kl) = -0.02256$ $\quad \ln(kl) = 1.144729$

$S_i(kl) = 1.851$ $\quad C_i(2ka^2/l) \simeq 0.577 + \ln(0.06283) \simeq 0.577 - 2.7672$

$= -2.190$

These yield $R_{im} \simeq 73.123$, $\quad R_{in} = 73.123$

$X_{im} \simeq 42.546$, $\quad X_{in} = 42.546$

c. $l = 3\lambda/4$: $kl = 3\pi/2$, $2kl = 3\pi$, $2ka^2/\lambda = 0.04188$, $kl/2 = 3\pi/4$

$C_i(kl) = -0.198$ $\quad \ln(kl/2) = 0.857$

$C_i(2kl) = 0.01071$ $\quad \ln(kl) = 1.5502$

$S_i(kl) = 1.6000$ $\quad C_i(2ka^2/\lambda) \simeq 0.577 + \ln(0.04188) \simeq 0.577 - 3.1729$

$S_i(2kl) = 1.6746$ $\quad = -2.596$

These yields $R_{im} \simeq 185.274$, $\quad R_{in} = 370.548$

$X_{im} \simeq 161.689$, $\quad X_{in} = 323.378$

d. $l = \lambda$: $kl = 2\pi$, $2kl = 2\pi$, $2ka^2/\lambda = 0.031415$, $kl/2 = \pi$

$C_i(kl) = -0.02256$ $\quad \ln(kl/2) = 1.144729$

$C_i(2kl) = -0.00612$ $\quad \ln(kl) = 1.837877$

$S_i(kl) = 1.4182$ $\quad C_i(2ka^2/\lambda) \simeq 0.577 + \ln(0.031415)$

$S_i(2kl) = 1.4925$ $\quad \simeq 0.577 - 3.4605 = -2.8837$

These yield $R_{im} \simeq 199.068$, $\quad R_{in} = 199.068/0 = \infty$

$X_{im} \simeq 125.409$, $\quad X_{in} = 125.409/0 = \infty$

8-10. The resonant length will be found using a "trial and error" method. A length will be assumed and the corresponding reactance will be found using the equation

$$X_{im} = 30\left\{2S_i(kl) + \cos(kl)\left[2S_i(kl) - S_i(2kl)\right] - \sin(kl)\left[2C_i(kl) - C_i(2kl) - C_i\left(\frac{2ka^2}{l}\right)\right]\right\}$$

The length for which $X_{im} \simeq 0$ will be the resonant length of the dipole.

a. Initially a $l = 0.95(0.5\lambda) = 0.475\lambda$ was assumed. For this

$$kl = 2.985, \quad 2kl = 5.969, \quad 2ka^2/\lambda\Big|_{a=\lambda/200} = 6.6138 \times 10^{-4}$$

$S_i(kl) = 1.848$ $\quad \cos(kl) = -0.9877$

$S_i(2kl) = 1.425$ $\quad \sin(kl) = 0.1564$

$C_i(kl) = 0.12$ $\quad C_i(2ka^2/\lambda) = C_{in}(2ka^2/\lambda) + 0.5772 + \ln(2ka^2/\lambda)$

$C_i(2kl) = -0.07$ $\quad \simeq 0 + 0.5772 - 7.32 = -6.743$

These lead to $X_{im} = 9.657 \neq 0$. For the next attempt let

$l = 0.93(0.5) = 0.465\lambda$, For this

$$kl = 2.9217, \quad 2kl = 5.8434, \quad 2ka^2/\lambda = 6.756 \times 10^{-4}$$

$S_i(kl) = 1.844$ $\quad \cos(kl) = -0.9759$

$S_i(2kl) = 1.434$ $\quad \sin(kl) = 0.2181$

$C_i(kl) = 0.135$ $\quad C_i(2ka^2/\lambda) = C_{in}(2ka^2/\lambda) + 0.5772 + \ln(2ka^2/\lambda)$

$C_i(2kl) = -0.092$ $\quad \simeq 0 + 0.5772 - 7.30 = -6.7228$

These lead to

$$X_{im} = 30\left\{2(1.844) - 0.9759\left[2(1.844) - 1.434\right] - 0.2181\left[2(0.135) + 0.092 - (-6.7228)\right]\right\}$$

$$X_{im} = -1.706 \simeq 0$$

We will take $l \simeq 0.465\lambda$ to be the approximate resonant length.

b. For $l = 0.465\lambda$

$$R_r = 60\left\{0.5772 + \ln(2.9217) - 0.135 + \tfrac{1}{2}(0.2181)\left[1.434 - 2(1.844)\right] + \tfrac{1}{2}(-0.9759)\left[0.5772 + \ln(1.46085) - 0.092 - 2(0.135)\right]\right\}$$

$$R_r = 60(0.9785) = 58.71 = R_{im}$$

c. $$R_{in} = \frac{R_{im}}{\sin^2(kl/2)} = \frac{58.71}{0.98796} = 59.42$$

d. $$\Gamma = \frac{R_{in} - Z_c}{R_{in} + Z_c} = \frac{59.42 - 50}{59.42 + 50} = 0.086$$

$$VSWR = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.086}{1 - 0.086} = 1.188$$

8-11 Using Fig. 8.17 and Fig 8.18

a. $a = 10^{-5}\lambda \Rightarrow \ell \simeq 0.478\lambda$, $R_r = R_{im} = 70$

b. $a = 10^{-4}\lambda \Rightarrow \ell \simeq 0.476\lambda$, $R_r = R_{im} \simeq 67.5$

c. $a = 10^{-3}\lambda \Rightarrow \ell \simeq 0.470\lambda$, $R_r = R_{im} = 64$

d. $a = 10^{-2}\lambda \Rightarrow \ell \simeq 0.441\lambda$, $R_r = R_{im} \simeq 57.5$

More accurate expressions can be obtained using equations (8-60) by trial and error, and (4-70).

8-12. $a = 10^{-2}\lambda$, $\ell = \lambda/4$

a. From Figure 4.8, equations (4-93) & (8-60)

$Z_{in} = 73 + j42.5$ for $\ell = \lambda/2$

Therefore for $\ell = \lambda/4$ above ground plane.

$$Z_{in} = \tfrac{1}{2}(73 + j42.5) = 36.5 + j21.25$$

b. From Table 9.2

$$F' = \frac{\ell/a}{1+\ell/a} = \frac{\lambda/4\,/10^{-2}\lambda}{1+\lambda/4/10^{-2}\lambda} = \frac{0.25\times10^{2}}{1+0.25\times10^{2}} = \frac{25}{26} = 0.9615$$

$$\ell' = 0.24\lambda F' = 0.24\,\lambda F' = 0.24\cdot\lambda\cdot(0.9615) = 0.2308\lambda$$

c. $Z_{in} \simeq 34$

d. $$\Gamma_{in} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} = \frac{34-50}{34+50} = -\frac{16}{84} = -0.1905$$

$$VSWR = \frac{1+|\Gamma_{in}|}{1-|\Gamma_{in}|} = \frac{1+0.1905}{1-0.1905} = 1.4706$$

8-13 a. For a side-by-side arrangement, the mutual impedance is obtained using (8-71a) - (8-71e), where (for $d = \lambda/4$)

$C_i(kd) = C_i(\pi/2) = 0.472$ $\quad S_i(kd) = S_i(\pi/2) = 1.37076$

$C_i(u_1) = C_i(2.118\pi) = 0.0334$ $\quad S_i(u_1) = S_i(2.118\pi) = 1.43$

$C_i(u_2) = C_i(0.118\pi) = -0.4534$ $\quad S_i(u_2) = S_i(0.118\pi) = 0.36696$

Thus $R_{21m} = 30[2(0.472) - 0.03334 + 0.4534] = 40.92$

$X_{21m} = -30[2(1.37076 - 1.43 - 0.36696] = -28.34$

Computer Result
R21l ⇒ 40.7857713
X21l ⇒ -28.34905
⇓
R21m = 40.7857713
X21m ⇒ -28.34905

b. For a collinear arrangement, the mutual impedance is obtained using (8-72a) - (8-72f) where (for $s = \lambda/4$) $\Rightarrow h = 3\lambda/4$

$C_i(2V_0) = C_i(3\pi) = 0.0106$ $\quad \sin(kh) = \sin(3\pi/2) = -1$, $\cos(kh) = 0$

$C_i(V_2) = C_i(\pi) = 0.07367$ $\quad S_i(2V_0) = S_i(3\pi) = 1.67476$

$C_i(V_1) = C_i(5\pi) = 0.00395$ $\quad S_i(V_2) = S_i(\pi) = 1.85193$

$\ln(V_3) = \ln(5/9) = -0.5878$ $\quad S_i(V_1) = S_i(5\pi) = 1.63396$

Thus $R_{21m} = 15(-1)[2(1.67476 - 1.85193 - 1.63396] = 2.0455$

$X_{21m} = 15(-1)[2(0.0106) - 0.07367 - 0.00395 + 0.5878] = -7.97$

Computer Result.
R2l = 2.04568
X2l = -7.970972
R21m = 2.04568
X21m = -7.9709

8-14. Using (8-64a)

$$Z_{1d} = \frac{V_1}{I_1} = Z_{11} + Z_{12}\left(\frac{I_2}{I_1}\right)$$

which for identical dipoles ($I_2 = I_1$) reduces to

$$Z_{1d} = Z_{11} + Z_{12}$$

For $S = 0.35\lambda$ we obtain from Figure 8.21 that

$$Z_{12} \simeq -2.5 - j6.0$$

Since for $\ell = \lambda/2$, $Z_{11} = 73 + j42.5$

$$Z_{1d} \simeq (73 - 2.5) + j(42.5 - 6) = 70.5 + j36.5$$

8-15. Examining the data of Figure 8.21 and knowing that the self impedance of a $\lambda/2$ dipole is $Z_{11} \simeq 73. + j42.5$, the driving impedance with the smallest reactance will occur when $S \simeq 0.375\lambda$ where $X_{21} \simeq -38.75$. Thus for $S \simeq 0.375\lambda$

$$Z_{1d} \simeq (73 - 23) + j(42.5 - 38.75) = 50 + j3.75$$

and the reactive part never gets smaller than that.

9-1. a. Using (9-12a) $\Rightarrow Z_c = 120 \ln[\cot(\frac{\alpha}{4})] = 300 \Rightarrow \alpha = 18.76°$

b. Using Fig. 9.4(b) and extrapolating

First: $\Rightarrow \pi \ell_1/\lambda \simeq 1.2 \Rightarrow \ell_1 \simeq \frac{1.2}{\pi}\lambda \simeq 0.382\lambda$

Second: $\Rightarrow \pi \ell_2/\lambda \simeq 2.8 \Rightarrow \ell_2 \simeq \frac{2.8}{\pi}\lambda \simeq 0.89\lambda$

c. Using Fig. 9.4(a)

$\pi\ell_1/\lambda = 1.2 : \Rightarrow R_{in} \simeq 25 \Rightarrow |\Gamma| = \left|\frac{25-300}{25+300}\right| = 0.846 \Rightarrow VSWR = \frac{1+0.846}{1-0.846} \simeq 12$

$\pi\ell_2/\lambda = 2.8 : \Rightarrow R_{in} \simeq 400 \Rightarrow |\Gamma| = \left|\frac{400-300}{400+300}\right| = 0.1428 \Rightarrow VSWR = \frac{1+0.1428}{1-0.1428} \simeq 1.33$

9-2. a. From Figure 9.8(b) From Figure 9.8(a)

$\ell_1(\ell_1/d_1 = 25) \simeq 0.45\lambda \Rightarrow d_1 = \frac{\ell_1}{25} = \frac{0.45\lambda}{25} = 1.8\times10^{-2}\lambda$ $R_{in}(\ell_1/d_1 = 25, \ell_1 = 0.45\lambda) \simeq 54$

$\ell_1(\ell_1/d_1 = 50) \simeq 0.45\lambda \Rightarrow d_1 = \frac{\ell_1}{50} = \frac{0.45}{50}\lambda = 9\times10^{-3}\lambda$ $R_{in}(\ell_1/d_1 = 50, \ell_1 = 0.45\lambda) \simeq 74$

$\ell_1(\ell_1/d_1 = 10^4) \simeq 0.49\lambda \Rightarrow d_1 = \frac{\ell_1}{10^4} = \frac{0.49}{10^4}\lambda = 4.9\times10^{-5}\lambda$ $R_{in}(\ell_1/d_1 = 10^4, \ell_1 = 0.45\lambda) \simeq 74$

$\ell_2(\ell_2/d_2 = 25) \simeq 0.75\lambda \Rightarrow d_2 = \frac{\ell_2}{25} = \frac{0.75}{25}\lambda = 3\times10^{-2}\lambda$ $R_{in}(\ell_2/d_2 = 25, \ell_2 = 0.48\lambda) \simeq 425$

$\ell_2(\ell_2/d_2 = 50) \simeq 0.80\lambda \Rightarrow d_2 = \frac{\ell_2}{50} = \frac{0.80}{50}\lambda = 1.6\times10^{-2}\lambda$ $R_{in}(\ell_2/d_2 = 50, \ell_2 = 0.80\lambda) \simeq 725$

$\ell_2(\ell_2/d_2 = 10^4) \simeq 1.03\lambda \Rightarrow d_2 = \frac{\ell_2}{10^4} = \frac{1.03}{10^4}\lambda = 1.03\times10^{-4}\lambda$ $R_{in}(\ell_2/d_2 = 10^4, \ell_2 = 1.03\lambda) \simeq 21,521.36$

b. From Table 9.1

$\ell_1(\ell_1/d_1 = 25) = 0.48\lambda F_1 \Rightarrow F_1 = \frac{\ell_1/d_1}{1+\ell_1/d_1} = \frac{25}{26} = 0.9615 \Rightarrow \ell_1(\ell_1/d_1 = 25) = 0.4615\lambda$

$d_1 = \frac{\ell_1}{25} = \frac{0.4615}{25}\lambda = 1.846\times10^{-2}\lambda \Rightarrow R_{in} = 67$

$\ell_1(\ell_1/d_1 = 50) = 0.48\lambda F_1 \Rightarrow F_1 = \frac{\ell_1/d_1}{1+\ell_1/d_1} = \frac{50}{51} = 0.9804 \Rightarrow \ell_1(\ell_1/d_1 = 50) = 0.4706\lambda$

$d_1 = \frac{\ell_1}{50} = \frac{0.4706}{50}\lambda = 9.41\times10^{-3}\lambda \Rightarrow R_{in} = 67$

$\ell_1(\ell_1/d_1 = 10^4) = 0.48\lambda F_1 \Rightarrow F_1 = \frac{\ell_1/d_1}{1+\ell_1/d_1} = \frac{10^4}{1+10^4} = 0.9999 \Rightarrow \ell_1(\ell_1/d_1 = 10^4) = 0.47995\lambda$

$d_1 = \frac{\ell_1}{10^4} = \frac{0.47995}{10^4}\lambda = 4.7995\times10^{-5}\lambda \Rightarrow R_{in} = 67$

$\ell_2(\ell_2/d_2 = 25) = 0.96\lambda F_2 \Rightarrow F_2 = \frac{\ell_2/d_2}{1+\ell_2/d_2} = \frac{25}{26} = 0.9615 \Rightarrow \ell_2(\ell_2/d_2 = 25) = 0.923\lambda$

$d_2 = \frac{\ell_2}{25} = \frac{0.923\lambda}{25} = 3.69\times10^{-2}\lambda$, $R_n = 150\log_{10}(\frac{\ell_2}{d_2}) = 150\log_{10}(25) = 209.69$

$R_{in} = R_n^2/67 \simeq (209.69)^2/67 = 656.27$

$\ell_2(\ell_2/d_2 = 50) = 0.96\lambda F_2 \Rightarrow F_2 = \frac{\ell_2/d_2}{1+\ell_2/d_2} = \frac{50}{51} = 0.9804 \Rightarrow \ell_2(\ell_2/d_2 = 50) = 0.9412\lambda$

$d_2 = \frac{\ell_2}{50} = \frac{0.9412\lambda}{50} = 1.88\times10^{-2}\lambda$, $R_n = 150\log_{10}(\frac{\ell_2}{d_2}) = 150\log_{10}(50) = 254.8455$

$R_{in} = R_n^2/67 = (254.8455)^2/67 = 969.35$

(Continued)

9-2 (Cont'd)

$\ell_2\,(\ell_2/d_2=10^4)=0.96\lambda F_2 \Rightarrow F_2=\dfrac{\ell_2/d_2}{1+\ell_2/d_2}=\dfrac{10^4}{1+10^4}=0.9999 \Rightarrow \ell_2\,(\ell_2/d_2=10^4)$

$\ell_2 = 0.9599\lambda$

$d_2=\dfrac{\ell_2}{10^4}=\dfrac{0.9599\lambda}{10^4}=9.599\times10^{-5}\lambda$

$R_n = 150\log_{10}\left(\dfrac{\ell_2}{d_2}\right)=150\log_{10}(10^4)=600$

$R_{in}=R_n^2/67=(600)^2/67=5,373.13$

9-3. From Table 9.2 $\Rightarrow F'=\left.\dfrac{\ell/a}{1+\ell/a}\right|_{\ell/d=50}=\left.\dfrac{2(\ell/d)}{1+2(\ell/d)}\right|_{\ell/d=50}=\dfrac{100}{1+100}=0.99$

$R_n'=\left.75\log_{10}(\ell/a)\right|_{\ell/d=50}=\left.75\log_{10}(2\ell/d)\right|_{\ell/d=50}=75\log_{10}(100)=150$

a. 1st Resonance: $\ell_1=0.24F'=0.23529\lambda$, $d_1=4.7\times10^{-3}\lambda$, $R_{in}=34$

b. 2nd Resonance: $\ell_2=0.48F'=0.47\lambda$, $d_2=9.4117\times10^{-3}\lambda$, $R_{in}=(150)^2/34=661.76$

c. 3rd Resonance: $\ell_3=0.72F'=0.706\lambda$, $d_3=14.1176\times10^{-3}\lambda$, $R_{in}=48$

d. 4th Resonance: $\ell_4=0.96F'=0.941176\lambda$, $d_4=18.823\times10^{-3}\lambda$, $R_{in}=(150)^2/48=468.75$

9-4. a. $\ell=\lambda/2$ (from (9-15))

$\ell/d=25 \Rightarrow Z_{in}=88.4+j27.5 \Rightarrow |\Gamma|=\left|\dfrac{88.4+j27.5-50}{88.4+j27.5+50}\right|=0.3347$

$VSWR=\dfrac{1+0.3347}{1-0.3347}=2.006$

$\ell/d=50 \Rightarrow Z_{in}=85.8+j54.9 \Rightarrow |\Gamma|=\left|\dfrac{85.8+j54.9-50}{85.8+j54.9+50}\right|=0.4474$

$VSWR=\dfrac{1+0.4474}{1-0.4474}=2.619$

$\ell/d=10^4 \Rightarrow Z_{in}=73+j42.5 \Rightarrow |\Gamma|=\left|\dfrac{73+j42.5-50}{73+j42.5+50}\right|=0.3713$

$VSWR=\dfrac{1+0.3713}{1-0.3713}=2.181$

b. $\ell=\lambda$ [From Figures 9.8(a) and 9.8(b)]

$\ell/d=25 \Rightarrow Z_{in}=180-j255 \Rightarrow |\Gamma|=\left|\dfrac{180-j255-50}{180-j255+50}\right|=0.8335$

$VSWR=\dfrac{1+0.8335}{1-0.8335}=11.012$

$\ell/d=50 \Rightarrow Z_{in}=280-j410 \Rightarrow |\Gamma|=\left|\dfrac{280-j410-50}{280-j410+50}\right|=0.8932$

$VSWR=\dfrac{1+0.8932}{1-0.8932}=17.729$

$\ell/d=10^4 \Rightarrow Z_{in}=(5.0487+j3.1804)\times10^4$ [Using Eqs. 4-70, 4-79, 8-60b, 8-61b]

$|\Gamma|\cong 1 \Rightarrow VSWR\simeq\dfrac{1+1}{1-1}=\infty$

(Continued)

9-4. (Cont'd)

c. $\ell = 3\lambda/2$ (from equation (9-15))

$\ell/d = 25 \Rightarrow Z_{in} = 106.8 + j4.9 \Rightarrow |\Gamma| = \left|\frac{106.8 + j4.9 - 50}{106.8 + j4.9 + 50}\right| = 0.3634$

$VSWR = \frac{1 + 0.3634}{1 - 0.3634} = 2.1417$

$\ell/d = 50 \Rightarrow Z_{in} = 103.3 + j9.2 \Rightarrow |\Gamma| = \left|\frac{103.3 + j9.2 - 50}{103.3 + j9.2 + 50}\right| = 0.3522$

$VSWR = \frac{1 + 0.3522}{1 - 0.3522} = 2.087$

$\ell/d = 10^4 \Rightarrow Z_{in} = 105.49 + j45.54 \Rightarrow |\Gamma| = \left|\frac{105.49 + j45.54 - 50}{105.49 + j45.54 + 50}\right| = 0.443$

$VSWR = \frac{1 + 0.443}{1 - 0.443} = 2.591$

9-5 Using Table 9.3

a. $a_e = 0.25\,a = 0.25(\lambda/10) = 0.025\,\lambda$

b. $a_e = 0.59\,a = 0.59(\lambda/10) = 0.059\,\lambda$

c. $b/a = 0.01\lambda/(0.1\lambda) = 1/10 = 0.1 \Rightarrow a_e = 0.2875\,a = 0.2875(0.1) = 0.02875$

d. $a_e = 0.5(a+b) = 0.5(\lambda/10 + \lambda/20) = 0.075\lambda$

e. Use Table 9.3 or (9-22a)

$a_e = \sqrt{as} = \sqrt{1.466 \times 10^{-2}(0.8)} = 0.1083$ cm

9-6 From (9-19) $\Rightarrow Z_0 = 120\cosh\left(\frac{S/2}{a}\right) = 120\ln\left[\frac{S/2 + \sqrt{(S/2)^2 - a^2}}{a}\right]$

a. $S = 6.13 \times 10^{-3}\lambda$, $d = 10^{-3}\lambda$, $a = 0.5 \times 10^{-3}\lambda$;

$Z_0 = 120\ln\left(\frac{3.065 + \sqrt{(3.065)^2 - (0.5)^2}}{0.5}\right) = 120\ln\left(\frac{3.065 + 3.0239}{0.5}\right) = 299.95$

b. $S = 2.13 \times 10^{-2}\lambda$, $d = 10^{-3}\lambda$, $a = 0.5 \times 10^{-3}\lambda$:

$Z_0 = 120\ln\left[\frac{10.65 + \sqrt{(10.65)^2 - (0.5)^2}}{0.5}\right] = 120\ln\left(\frac{10.65 + 10.63825}{0.5}\right) = 450.156$

c. $S = 7.42 \times 10^{-2}\lambda$, $d = 10^{-3}\lambda$, $a = 0.5 \times 10^{-3}\lambda$

$Z_0 = 120\ln\left(\frac{37.1 + \sqrt{(37.1)^2 - (0.5)^2}}{0.5}\right) = 120\ln\left(\frac{37.1 + 37.0966}{0.5}\right) = 599.984$

9-7. From Table 9.3

$$\ln a_e \simeq \frac{1}{(S_1 + S_2)^2}\left[S_1^2 \ln a_1 + S_2^2 \ln a_2 + 2S_1 S_2 \ln S\right]$$

If both conductors are circular $\Rightarrow S_1 = 2\pi a_1$, $S_2 = 2\pi a_2$. Thus

$$\ln a_e \simeq \frac{1}{(a_1 + a_2)^2}\left[a_1^2 \ln a_1 + a_2^2 \ln a_2 + 2a_1 a_2 \ln S\right]$$

Expanding the denominator and dividing numerator & denominator by $a_1^2$ reduces to

(Continued)

9-7 (Cont'd)

$$\ln a_e \simeq \frac{1}{1+2\frac{a_2}{a_1}+\left(\frac{a_2}{a_1}\right)^2}\left[\ln a_1+\left(\frac{a_2}{a_1}\right)^2 \ln a_2+2\left(\frac{a_2}{a_1}\right)\ln S\right]$$

Defining $u = a_2/a_1$, $v = S/a_1$, we can write that

$$\ln a_e \simeq \frac{1}{1+2u+u^2}\left[\ln a_1+u^2\ln(ua_1)+2u\ln(va_1)\right]$$

$$\simeq \frac{1}{1+2u+u^2}\left[\ln a_1+u^2\ln u+u^2\ln a_1+2u\cdot\ln v+2u\cdot\ln a_1\right]$$

$$\simeq \frac{1}{(1+u)^2}\left[(u^2\ln a_1+2u\ln a_1+\ln a_1)+u^2\ln u+2u\ln v\right]$$

$$\simeq \frac{1}{(1+u)^2}\left[(1+u)^2\ln a_1+u^2\ln u+2u\ln v\right]$$

$$\ln a_e \simeq \ln(a_1)+\frac{1}{(1+u)^2}\left[u^2\ln u+2u\cdot\ln v\right]$$

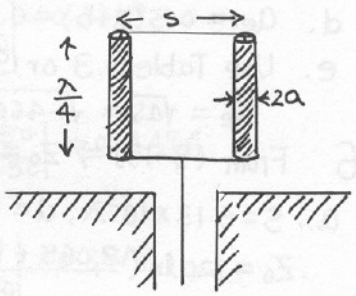
9-8 $\sigma = 5.7\times10^7$ S/m  $f = 1$ GHz $\Rightarrow \lambda = \frac{3\times10^8}{10^9} = 0.3$ m

$a = \lambda/200$, $S = \lambda/50$

a. $a_e = \sqrt{aS} = \sqrt{\frac{\lambda^2}{50(200)}} = \sqrt{\frac{\lambda^2}{10{,}000}} = \frac{\lambda}{100} = 3\times10^{-3}$ m

$a = \lambda/200 = 1.5\times10^{-3}$ m

$a_e/a = (\lambda/100)/(\lambda/200) = 2$




b. $R_{hf} = \frac{l}{P}\sqrt{\frac{\omega\mu_0}{2\sigma}} = \frac{\lambda/4}{2\pi a}\sqrt{\frac{2\pi(10^9)(4\pi\times10^{-7})}{2\cdot(5.7\times10^7)}}$

$= \frac{\frac{\lambda}{4}}{2\pi\left(\frac{\lambda}{200}\right)}\sqrt{\frac{10^2 4\pi^2}{5.7\times10^7}} = \frac{100}{4\pi}\frac{2\pi(10)}{\sqrt{57}}\times10^{-3}$

$R_{hf} = \frac{50}{\sqrt{57}}\times10^{-2} = 6.6227\times10^{-2} = 0.06627$

$R_{hf}$ (single) = 0.06627 ohms

$R_{hf}$ (double) = $\frac{1}{2}R_{hf}$ (Single) = 0.03313 ohms

$R_{hf}$ (effective) = $\frac{1}{2}R_{hf}$ (single) = 0.03313 ohms

C. $e_{cd} = \frac{R_r}{R_r+R_L}$,  $R_r = \frac{1}{2}(173) = 36.5$

$e_{cd}$ (single) = $\frac{36.5}{36.5+0.06627}\times100 = 99.82\%$

$e_{cd}$ (double) = $\frac{36.5\times100}{36.5+0.03313} = 99.909\%$

$e_{cd}$ (effective) = $\frac{36.5\times100}{36.5+0.03313} = 99.909\%$

9-9. Using (9-18) with $\ell' = \lambda/2$ ($\ell = \lambda/4$), we have that

$$Z_t = j Z_0 \cdot \tan\left(\frac{2\pi}{\lambda}\frac{\lambda}{4}\right) = j Z_0 \tan\left(\frac{\pi}{2}\right) = \infty$$

Thus (9-24) reduces to

$$Z_{in} = \left.\frac{4 Z_t Z_d}{2Z_d + Z_t}\right|_{Z_t=\infty} = \left.\frac{4Z_d}{2\frac{Z_d}{Z_t}+1}\right|_{Z_t=\infty} = 4Z_d$$

9-10. $d = 10^{-3}\lambda$, $S = 6.13\times10^{-3}\lambda$

a. From the solution of Prob 9.6(a) $\Rightarrow Z_0 = 299.95 \simeq 300$

Using Figure 9.15 $\Rightarrow \ell \simeq 0.325\lambda$ : $Z_{in} = R_{in} + jX_{in} \simeq 1,700$

b. $|\Gamma| = \left|\frac{1,700-300}{1,700+300}\right| = 0.7 \Rightarrow \text{VSWR} = \frac{1+0.7}{1-0.7} = 5.667$

9-11. $\ell/d = 500$, $S = 6.13\times10^{-3}\lambda \Rightarrow S/2 = 3.065\times10^{-3}\lambda$

a. Using Table 9.1

$$\ell = 0.48F\lambda = 0.48\lambda\frac{\ell/d}{1+\ell/d} = 0.48\lambda\frac{500}{1+500} = 0.48\lambda(0.998) = 0.479\lambda$$

b. $\ell/d = 500 \Rightarrow d = \frac{\ell}{500} = \frac{0.479\lambda}{500} = 0.958\times10^{-3}\lambda$, $a = 0.479\times10^{-3}\lambda$

c. Using (9-19) $\Rightarrow Z_0 = 120\ln\left[\frac{S/2+\sqrt{(S/2)^2-a^2}}{a}\right] = 120\ln\left[\frac{3.065+\sqrt{(3.065)^2-(0.479)^2}}{0.479}\right]$

$$Z_0 = 120\ln\left(\frac{3.065+3.027}{0.479}\right) = 305.17$$

d. Using (9-18) with $\ell = 0.479\lambda$

$$Z_t = j Z_0 \tan\left(k\frac{\ell}{2}\right)\Big|_{\ell=0.479\lambda} = j(305.17)\tan\left(\frac{2\pi}{\lambda}0.2395\lambda\right) = j4,618.93$$

e. Using (9-24) $\Rightarrow Z_{in} = \frac{4Z_t Z_d}{2Z_d + Z_t}$

$Z_d = R_d + jX_d$ where according to (4-70, 8-60, 8-61)

$$R_r = 60\{C + \ln(k\ell) - C_i(k\ell) + \tfrac{1}{2}\sin(k\ell)[S_i(2k\ell) - 2S_i(k\ell)] + \tfrac{1}{2}\cos(k\ell)[C + \ln(\tfrac{k\ell}{2}) + C_i(2k\ell) - 2C_i(k\ell)]\}$$

$$X_m = 30\{2S_i(k\ell) + \cos(k\ell)[2S_i(k\ell) - S_i(2k\ell)] - \sin(k\ell)[2C_i(k\ell) - C_i(2k\ell) - C_i(\tfrac{2ka^2}{\ell})]\}$$

$C = 0.5772$, $R_d = \frac{R_r}{\sin^2(k\ell/2)}$, $X_d = \frac{X_m}{\sin^2(k\ell/2)}$

1. $a = 0.479\times10^{-3}\lambda$, $\ell = 0.479\lambda$

$k\ell = 3.01$, $2k\ell = 6.0129 = 6.02$, $k\ell/2 = 1.5048$

$2ka^2/\ell = 6.019\times10^{-6}$

$C_{in}(x) = \ln(\gamma x) - C_i(x) = \ln(\gamma) + \ln(x) - C_i(x) = 0.5772 + \ln x - C_i(x)$

$C_{in}(6.019\times10^{-6}) \simeq 0 = 0.5772 + \ln(6.019\times10^{-6}) - C_i(6.019\times10^{-6})$

(Continued)

9-11 (Cont'd)

$$C_i(2ka^2/\ell) = C_i(6.019\times10^{-6}) = 0.5772 - 12.0205 = -11.443$$

$C_i(k\ell) = C_i(3.01) = 0.115$
$S_i(k\ell) = S_i(3.01) = 1.849$
$C_i(2k\ell) = C_i(6.019) = -0.068$
$S_i(2k\ell) = S_i(6.019) = 1.424$
} From Appendix III

$\ln(k\ell) = \ln(3.01) = 1.102$
$\ln(\frac{k\ell}{2}) = \ln(1.505) = 0.409$
$\sin(k\ell) = \sin(3.01) = 0.132$
$\cos(k\ell) = \cos(3.01) = -0.991$

$$R_r = 60\{0.5772 + 1.102 - 0.115 + \frac{0.132}{2}[1.424 - 2(1.849)] - \frac{0.991}{2}[0.5772 + 0.409 - 0.068 - 2(0.115)]\} = 60(1.073) = 64.380$$

$$X_m = 30\{2(1.849) - 0.991[2(1.849) - 1.424] - 0.132[2(0.115) - (-0.068) - (-11.443)]\}$$

$$X_m = 30(-0.1058) = -3.176$$

$$Z_d = R_d + jX_d = \frac{64.380 - j3.176}{\sin^2(1.5048)} = 1.004(64.38 - j3.176) = 64.6375 - j3.1887$$

Using (9-24)

$$Z_{in} = \frac{4Z_t Z_d}{2Z_d + Z_t} = \frac{4(4,618.93)(64.6375 - j3.1887)}{2(64.6375 - j3.1887) + 4,618.93} = 259.08 - j5.472$$

2. $a_e = \sqrt{as} = \sqrt{6.13\times10^{-3}\lambda(0.479\times10^{-3}\lambda)} = 1.714\times10^{-3}\lambda$, $\ell = 0.479\lambda$

Using the equivalent radius $a_e$ as the radius

$$2ka_e^2/\ell = 2(\tfrac{2\pi}{\lambda})(1.714\times10^{-3}\lambda)^2/(0.479\lambda) = 7.707\times10^{-5}$$

$$C_{in}(x) \simeq 0 = 0.5772 + \ln(7.707\times10^{-5}) - C_i(7.707\times10^{-5}),\quad x = 7.707\times10^{-5}$$

$$C_i(2ka_e^2/\ell) = C_i(7.707\times10^{-5}) + 0.5772 = -8.8936$$

$X_m \simeq 7.0155$, $R_r = 64.380$

$$Z_d = 1.004(64.380 + j7.0155) = 64.6375 + j7.04356$$

$$Z_{in} = \frac{4Z_t Z_d}{2Z_d + Z_t} = \frac{4(4,618.93)(64.6375 + j7.04356)}{2(64.6375 + j7.04356) + 4,618.93} = 251.59 + j26.665$$

3. Using $a = S/2 = 3.065\times10^{-3}\lambda$, $\ell = 0.479\lambda$

$$2a^2k/\ell = 2(3.065\times10^{-3}\lambda)^2(\tfrac{2\pi}{\lambda})/(0.479\lambda) = 2.4645\times10^{-4}$$

$$C_{in}(2.4645\times10^{-4}) \simeq 0 = \ln(2.4645\times10^{-4}) - C_i(2.4645\times10^{-4}) + 0.5772$$

$$C_i(2ka^2/\ell) = C_i(2.4645\times10^{-4}) = 0.5772 + \ln(2.4645\times10^{-4}) = -7.7311$$

$X_m \simeq 11.6$, $R_r = 64.380$

$$Z_d = 1.004(64.380 + j11.6) = 64.6375 + j11.6464$$

$$Z_{in} = \frac{4Z_t Z_d}{2Z_d + Z_t} = \frac{4(4,618.93)(64.6375 + j11.6464)}{2(64.6375 + j11.6464) + 4,618.93} = 251.556 + j43.76$$

9-12. $Z_{in} = 315.14 \angle 13.829°$

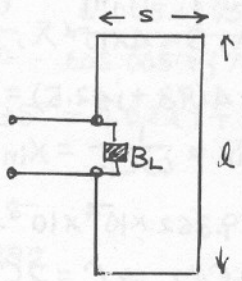
$= 306.005 + j75.326$

$$Y_{in} = \frac{1}{315.14 \angle 13.829°} = 3.173 \times 10^{-3} \angle -13.829°$$

$Y_{in} = 3.081 \times 10^{-3} - j0.758 \times 10^{-3} = (3.081 - j0.758) \times 10^{-3}$

$Y_{in} = G_{in} + jB_{in}$, $G_{in} = 3.081 \times 10^{-3}$

$B_{in} = -0.758 \times 10^{-3}$

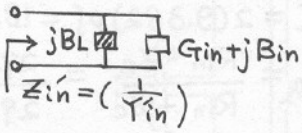



a. $B_L = -B_{in} = 0.758 \times 10^{-3}$

Capacitor

b. $B_L = 0.758 \times 10^{-3} = \omega C_L = 2\pi f C_L$

$$C_L = \frac{0.758 \times 10^{-3}}{2\pi f} = \frac{0.758 \times 10^{-3}}{2\pi (10 \times 10^6)} = 12.064 \times 10^{-12} = 12.064 \text{ pf.}$$




c. $Y_{in}' = 3.081 \times 10^{-3} = Y_{in} + Y_L$

$$Z_{in}' = \frac{1}{3.081 \times 10^{-3}} = 0.32457 \times 10^{3} = 324.57$$

$Z_{in}' \cong 324.57$ $(Z_{in}' = R_{in}')$

d. $$\Gamma_{in} = \frac{Z_{in}' - Z_o}{Z_{in}' + Z_o} = \frac{324.57 - 300}{324.57 + 300}$$

$$\Gamma_{in} = \frac{24.57}{624.57} = 0.03934$$

$$VSWR = \frac{1 + |\Gamma_{in}|}{1 - |\Gamma_{in}|} = \frac{1 + 0.03934}{1 - 0.03934}$$

$$VSWR = \frac{1.03934}{0.96066} = 1.082$$

9-13. $\ell = \lambda/2$, $Z_c = 300$

a. $Z_{in} = 4(73. + j42.5) = 292 + j170.$ $Y_{in} = \frac{1}{Z_{in}} = \frac{1}{292 + j170} \cdot \frac{292 - j170}{292 - j170} = \frac{292 - j170}{114,164}$

$Y_{in} = 2.5577 \times 10^{-3} - j1.489 \times 10^{-3} = (2.5577 - j1.489) \times 10^{-3} = G_{in} - jB_{in}$

$B_{in} = \omega C_{in} = 2\pi f C_{in} = B_{in} \Rightarrow C_{in} = \frac{B_{in}}{2\pi f} = \frac{1.489 \times 10^{-3}}{2\pi \cdot (108)} = 2.370 \times 10^{-12} f$

$C_{in} = 2370 \times 10^{-12} f = 2.37 \text{ pf}$

(Continued)

9-13 (Cont'd)

b. $G_{in} = 2.5577 \times 10^{-3}$

$$R_{in} = \frac{1}{G_{in}} = \frac{1}{2.5577 \times 10^{-3}} = 0.390976 \times 10^{3} = 390.976$$

$$\Gamma_{in} = \frac{R_{in} - Z_c}{R_{in} + Z_c} = \frac{390.976 - 300}{390.976 + 300} = 0.13166$$

$$VSWR = \frac{1 + |\Gamma_{in}|}{1 - |\Gamma_{in}|} = \frac{1 + 0.13166}{1 - 0.13166} = 1.303$$

9-14. $a = 10^{-3}\lambda$, $S = 4 \times 10^{-3}\lambda$, $Z_c = 300$

a. $Z_{in} = 4(73 + j42.5) = 292 + j170 = R_{in} + jX_{in}$

$$X_c = X_{in} = \frac{1}{\omega C_{in}} = X_{in} \Rightarrow C_{in} = \frac{1}{\omega X_{in}} = \frac{1}{2\pi f X_{in}} = \frac{1}{2\pi(10^8)(170)}$$

$$C_{in} = 9.362 \times 10^{-4} \times 10^{-8} = 9.362 \times 10^{-12} f = 9.362\ pf$$

$$C_{in} = C/2 \Rightarrow C = 2C_{in}$$

$$C = 2(9.362)\ pf = 18.724\ pf$$

b. $$\Gamma_{in} = \frac{R_{in} - Z_c}{R_{in} + Z_c} = \frac{292 - 300}{292 + 300} = \frac{-8}{592} = -0.0135$$

$$VSWR = \frac{1 + |\Gamma_{in}|}{1 - |\Gamma_{in}|} = \frac{1 + 0.0135}{1 - 0.0135} = 1.0274$$

9-15. $\ell = 0.47\lambda$, $a = 0.005\lambda$, $S = 0.025\lambda$

$$Z_t = jZ_o \tan\left(k\frac{\ell}{2}\right)\Big|_{\ell = 0.47\lambda}$$

$$Z_o = 120 \cosh^{-1}\left(\frac{S/2}{a}\right) = 120 \cosh^{-1}\left(\frac{0.025}{2 \times 0.005}\right)$$

$$= 120 \cosh^{-1}(2.5) = 120(1.5667) = 188.016$$

$$Z_o \simeq 188$$

$$Z_t = j188 \tan\left[\frac{2\pi}{\lambda}\frac{0.47\lambda}{2}\right] = j188 \tan(0.47\pi)$$

$$= j188(10.579) = j\,1,988.83$$

$$Z_d = 79 + j13$$

a. Determine the:

(i). input impedance: $Z_{in} = \dfrac{4Z_t Z_d}{2Z_d + Z_t} = \dfrac{4(j\,1,988.83)(79 + j13)}{2(79 + j13) + j\,1,988.83}$

$$Z_{in} = j4(1,988.83)\frac{79 + j13}{158 + j26 + j1,988.83} = j4(1,988.83)\frac{80.06\angle 9.345^\circ}{158 + j\,2,014.83}$$

$$= j4(1,988.83)\frac{80.06\angle 9.345^\circ}{2,021.0155\angle 85.516}$$

$$Z_{in} = 315.14\ \angle 13.829^\circ = 306.005 + j75.326$$

(Continued)

9-15 (Cont'd)

(ii) $Z_{in} = 306.005 + j75.326$

$Z_d = 79 + j13$

$$\frac{R_{in}}{R_{din}} = \frac{306.005}{79} = 3.8735 \,, \quad \frac{X_{in}}{X_{din}} = \frac{75.326}{13} = 5.794$$

(iii) $$\Gamma_{in} = \frac{Z_{in} - Z_c}{Z_{in} + Z_c} = \frac{306.005 + j75.326 - 300}{306.005 + j75.326 + 300} = \frac{6.005 + j75.326}{606.005 + j75.326}$$

$$= \frac{75.366 \angle 85.44^\circ}{610.668 \angle 7.085^\circ} = 0.1234 \angle 78.35^\circ = 0.0249 + j0.1208$$

$|\Gamma_{in}| = 0.1234$

(iv) input VSWR : $$VSWR = \frac{1 + 0.1234}{1 - 0.1234} = 1.282$$

b. (i) $Z_{in} = 306.005 + j75.326$

$Z_{in}' = Z_{in} + Z_L = 306.005$

$Z_L = R_L + jX_L = -j75.326$

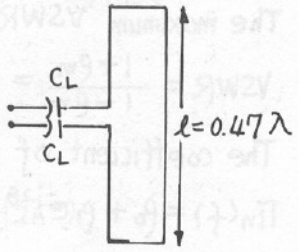
$$|X_L| = 75.326 = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

$$C = \frac{1}{2\pi f |X_L|} = \frac{1}{2\pi (10 \times 10^6)(75.326)}$$

$C = 211.288 \times 10^{-12} = 211.288$ pF

$$C = \frac{C_L^2}{2C_L} = \frac{C_L}{2} \Rightarrow C_L = 2C$$

$C_L = 2(211.288 \times 10^{-12})$, $C_L = 422.576$ pF




(ii) $Z_{in}' = 306.005$

$$\Gamma_{in} = \frac{Z_{in}' - Z_o}{Z_{in}' + Z_o} = \frac{306.005 - 300}{306.005 + 300} = \frac{6.005}{606.005} = 0.009909 \simeq 0.01$$

$\Gamma_{in} \simeq 0.01$

(iii) $$VSWR = \frac{1 + |\Gamma_{in}|}{1 - |\Gamma_{in}|} = \frac{1 + 0.01}{1 - 0.01} = \frac{1.01}{0.99}$$

$VSWR = 1.02$

9-16. $\ell = \lambda/2 \Rightarrow Z_{in} = Z_s = Z_m = 73 + j42.5$

a. Using (9-32) $\Rightarrow Z_{as} \simeq \dfrac{Z_s}{\cos^2(k\Delta\ell)} = \dfrac{73+j42.5}{\cos^2(\frac{2\pi}{\lambda}\cdot\frac{\lambda}{8})} = 146 + j85$

b. Using (4-77a) $\Rightarrow Z_{in}(z'=\frac{\lambda}{8}) = \left[\dfrac{I_0}{I_{in}(z'=\frac{\lambda}{8})}\right] Z_m = \left\{\dfrac{I_0}{I_0 \sin[k(\frac{\ell}{2}-z')]}\right\}^2 \cdot Z_m$ $\ell=\lambda/2,\ z'=\lambda/8$

$$Z_{in}(z'=\tfrac{\lambda}{8}) = \frac{Z_m}{\sin^2(k\frac{\lambda}{8})} = \frac{Z_m}{\sin^2(\frac{\pi}{4})} = 146 + j85$$

9-17 $N=2,\ R_L=\infty,\ Z_0=50$

Using (9-39) $\Rightarrow \dfrac{\Delta f}{f_0} = 2(1-\frac{2}{\pi}\theta_m) = 0.375 \Rightarrow \theta_m = 1.2763$ radians $(73.126°)$

$\sec\theta_m = \sec(73.126°) = 3.445$

Using (9-45) $\Rightarrow T_2(\sec\theta_m) = 2(\sec\theta_m)^2 - 1 = 2(3.445)^2 - 1 = 22.736$

From (9-44) $\Rightarrow \rho_m = \left|\dfrac{Z_L - Z_0}{Z_L + Z_0}\ \dfrac{1}{T_2(\sec\theta_m)}\right| = \left|\dfrac{(100-50)}{(100+50)}\ \dfrac{1}{22.736}\right| = 0.01466$

The maximum VSWR is

$$VSWR = \frac{1+\rho_m}{1-\rho_m} = \frac{1+0.01466}{1-0.01466} \simeq 1.0298$$

The coefficient of (9-37) can be written as

$$\Gamma_{in}(f) = \rho_0 + \rho_1 e^{-j2\theta} + \rho_2 e^{-j4\theta} = e^{-j2\theta}\left[(\rho_0 e^{j2\theta} + \rho_2 e^{-j2\theta}) + \rho_1\right]$$

whose magnitude, for $\rho_0 = \rho_2$, reduces to [when equated to (9-43) & (9-44)]

$$\rho_{in}(f) = 2(\rho_0 \cos 2\theta + \rho_1/2) = \rho_m T_2(\sec\theta_m \cos\theta) = \rho_m[\sec^2\theta_m \cos 2\theta + (\sec^2\theta_m - 1)]$$

by using (9-45). Equating similar terms

$$\Rightarrow \begin{cases} \rho_0 = \dfrac{\rho_m}{2}\sec^2\theta_m = \dfrac{0.01466}{2}(3.445)^2 = 0.087 \\ \rho_1 = \rho_m(\sec^2\theta_m - 1) = 0.01466\,[(3.445)^2 - 1] = 0.159 \end{cases}$$

$$Z_1 = \left(\frac{1+\rho_0}{1-\rho_0}\right) Z_0 = \left(\frac{1+0.087}{1-0.087}\right) 50 = 59.93$$

$$Z_2 = \left(\frac{1+\rho_1}{1-\rho_1}\right) Z_1 = \left(\frac{1+0.159}{1-0.159}\right) 59.93 \simeq 82.04$$

9-18. $N=3,\ R_L=100,\ Z_0=50$

a. $\rho_n = 2^{-N}\dfrac{R_L-Z_0}{R_L+Z_0}\dfrac{N!}{(N-n)!\,n!}$

$n=0$: $\rho_0 = \dfrac{1}{2^3}\dfrac{50}{150}\dfrac{3!}{3!\,0!} = \dfrac{1}{24} = \dfrac{Z_1-Z_0}{Z_1+Z_0} \Rightarrow Z_1 = 54.35$

$n=1$: $\rho_1 = \dfrac{1}{2^3}\dfrac{50}{150}\cdot\dfrac{3!}{2!\,1!} = \dfrac{1}{8} = \dfrac{Z_2-Z_1}{Z_2+Z_1} \Rightarrow Z_2 = 69.88$

$n=2$: $\rho_2 = \dfrac{1}{2^3}\dfrac{50}{150}\cdot\dfrac{3!}{1!\,2!} = \dfrac{1}{8} = \dfrac{Z_3-Z_2}{Z_3+Z_2} \Rightarrow Z_3 = 89.84$

$n=3$: $\rho_3 = \dfrac{1}{2^3}\cdot\dfrac{50}{150}\cdot\dfrac{3!}{0!\,3!} = \dfrac{1}{24}$

For a fractional bandwidth of 0.375, the maximum allowable reflection coefficient is equal to

$$\frac{\Delta f}{f_0} = 0.375 = 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{\rho_m}{(R_L-Z_0)/(R_L+Z_0)}\right]^{1/N} = 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{\rho_m}{50/150}\right]^{1/3}$$

$$\rho_m = 0.00815$$

and the VSWR is equal to

$$VSWR = \frac{1+\rho_m}{1-\rho_m} = \frac{1+0.00815}{1-0.00815} = 1.0164$$

b. $\dfrac{\Delta f}{f_0} = 0.375 = 2\left(1-\dfrac{2}{\pi}\theta_m\right) \Rightarrow \theta_m = 1.2763$ radians $(73.126^\circ)$

$\sec\theta_m = \sec(73.126^\circ) = 3.445$

For a 3-section transformer (N=3), using (9-45)

$$T_3(\sec\theta_m) = 4(\sec\theta_m)^3 - 3\sec\theta_m = 4(3.445)^3 - 3(3.445) = 153.2$$

Using (9-44) $\Rightarrow \rho_m = \left|\dfrac{Z_L-Z_0}{Z_L+Z_0}\dfrac{1}{T_N(\sec\theta_m)}\right| = \left|\dfrac{100-50}{100+50}\dfrac{1}{T_3(\sec\theta_m)}\right| = 0.002176$

$$VSWR = \frac{1+\rho_m}{1-\rho_m} = \frac{1+0.002176}{1-0.002176} = 1.00436$$

The coefficient of (9-37) can be written as

$$\Gamma_{in}(f) = \rho_0 + \rho_1 e^{-j2\theta} + \rho_2 e^{-j4\theta} + \rho_3 e^{-j6\theta} = e^{-j3\theta}\left(\rho_0 e^{j3\theta} + \rho_1 e^{j\theta} + \rho_2 e^{-j\theta} + \rho_3 e^{-j3\theta}\right)$$

which for $\rho_0 = \rho_3$ and $\rho_1 = \rho_2$ reduces to

$$\Gamma_{in}(f) = 2e^{-j3\theta}\left(\rho_0\cos 3\theta + \rho_1\cos\theta\right)$$

whose magnitude is equal to

$$\rho_{in}(f) = |\Gamma_{in}(f)| = 2\left(\rho_0\cos 3\theta + \rho_1\cos\theta\right)$$

when equated to (9-43), using (9-44), we can write that

(Continued)

9-18. (Cont'd)

$$2(\rho_0 \cos 3\theta + \rho_1 \cos\theta) = \rho_m T_3(\sec\theta_m \cdot \cos\theta)$$

$$2(\rho_0 \cos 3\theta + \rho_1 \cos\theta) = \rho_m [\sec^3\theta_m \cos 3\theta + 3(\sec^3\theta_m - \sec\theta_m)\cos\theta]$$

Equating like terms

$$2\rho_0 \cos 3\theta = \rho_m \sec^3\theta_m \cos 3\theta \Rightarrow \rho_0 = \rho_m \frac{\sec^3\theta_m}{2} = 0.002176 \frac{(3.445)^3}{2} = 0.04448$$

$$2\rho_1 \cos\theta = 3\rho_m(\sec^3\theta_m - \sec\theta_m)\cos\theta \Rightarrow \rho_1 = \rho_m \frac{\sec^3\theta_m - \sec\theta_m}{2} = 0.1222$$

Thus $\rho_0 = \rho_3 = 0.04448$

$\rho_1 = \rho_2 = 0.1222$

$$\rho_0 = 0.04448 = \frac{Z_1 - Z_0}{Z_1 + Z_0} = \frac{Z_1 - 50}{Z_1 + 50} \Rightarrow Z_1 = 54.655$$

$$\rho_1 = 0.1222 = \frac{Z_2 - Z_1}{Z_2 + Z_1} = \frac{Z_2 - 54.655}{Z_2 + 54.655} \Rightarrow Z_2 = 69.872$$

$$\rho_2 = 0.1222 = \frac{Z_3 - Z_2}{Z_3 + Z_2} = \frac{Z_3 - 69.872}{Z_3 + 69.872} \Rightarrow Z_3 = 89.326$$

9-19. $Z_0 = 300$, $a = 10^{-3}\lambda$, $N = 3$

a. From Table 9.1

$$F = \frac{1/2a}{1 + 1/2a} = \frac{0.5\lambda/(2\times10^{-3}\lambda)}{1 + 0.5\lambda/(2\times10^{-3}\lambda)} = \frac{0.25\times10^3}{1 + 0.25\times10^3} = \frac{250}{251} = 0.9960$$

$$\ell' = 0.48\lambda F = 0.9960\lambda(0.48) = 0.4781\lambda$$

$R_{in} \simeq 67$ ohms $= R_L$

$$\rho_n = Z^{-N}\frac{R_L - Z_0}{R_L + Z_0} C_n^N = Z^{-3}\frac{67 - 300}{67 + 300} \cdot C_n^N = \frac{1}{8}(-0.63488) C_n^N = -0.07936\, C_n^N$$

$$C_n^N = \frac{N!}{(N-n)!\,n!}, \quad \rho_n = -0.07936\, C_n^N$$

$n=0$: $C_0^3 = \frac{3!}{3!\,0!} = 1 \Rightarrow \rho_0 = -0.07936 \cdot 1 = -0.07936 = \frac{Z_1 - Z_0}{Z_1 + Z_0}$

$$Z_1 = Z_0\frac{1+\rho_0}{1-\rho_0} = 300\left(\frac{1 - 0.07936}{1 + 0.07936}\right) = 300(0.85295) = 255.88$$

$n=1$: $C_1^3 = \frac{3!}{2!\,1!} = 3 \Rightarrow \rho_1 = -0.07936\,(3) = -0.2381 = \frac{Z_2 - Z_1}{Z_2 + Z_1}$

$$Z_2 = Z_1\frac{1+\rho_1}{1-\rho_1} = 255.88\left(\frac{1 - 0.2381}{1 + 0.2381}\right) = 255.88(0.654) = 157.46$$

$n=2$: $C_2^3 = \frac{3!}{2!\,1!} = 3 \Rightarrow \rho_2 = -0.07936\,(3) = -0.2381 = \frac{Z_3 - Z_2}{Z_3 + Z_2}$

$$Z_3 = Z_2\cdot\frac{1+\rho_2}{1-\rho_2} = 157.46\left(\frac{1 - 0.2381}{1 + 0.2381}\right) = 157.46\,(0.6154) = 96.90$$

$n=3$: $C_3^3 = \frac{3!}{0!\,3!} = 1 \Rightarrow \rho_3 = -0.07936 \cdot 1 = -0.07936 = \frac{Z_L - Z_3}{Z_L + Z_3}$

$$Z_L = Z_3\cdot\frac{1+\rho_3}{1-\rho_3} = 96.90\left(\frac{1 - 0.07936}{1 + 0.07936}\right) = 96.90\,(0.85295) = 82.65$$

9-20. a) $L = 0.4665\lambda$

$R_{in} = 59.98\,\Omega$

b) $Z_1 = 70.84\,\Omega$

$Z_2 = 63.51\,\Omega$

c) See Fig. P9-20.

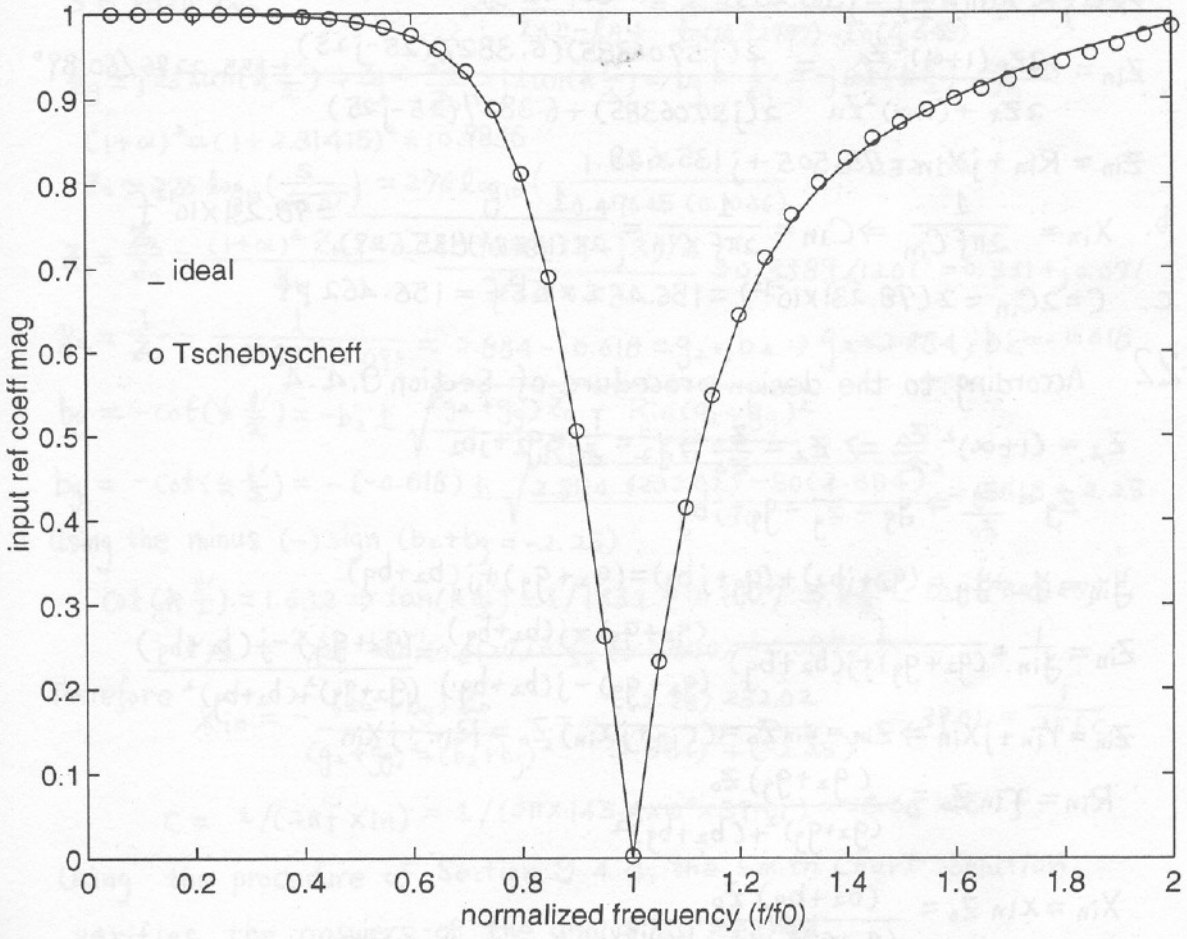



Fig P9-20

9-21. $f=15\text{MHz} \Rightarrow \lambda=\frac{300\times10^6}{15\times10^6}=20\text{ m}=2000\text{ cm}$

$d=1.9\times10^{-3}\lambda=3.8\text{ cm} \Rightarrow a=0.95\times10^{-3}\lambda=1.9\text{ cm}$

$d'=6.35\times10^{-4}\lambda=1.27\text{cm} \Rightarrow a'=3.175\times10^{-4}\lambda=0.635\text{ cm}$

$S=7.62\times10^{-3}\lambda=15.24\text{cm}$

$l'/2=0.285\lambda=57\text{cm}$, $Z_a=25-j25$

a. $u=a/a'=1.9/0.635=2.9921$

$v=S/a'=15.24/0.635=24.$

$$\alpha\simeq\frac{\ln v}{\ln v-\ln u}=\frac{\ln(24)}{\ln(24)-\ln(2.9921)}=1.5264$$

$(1+\alpha)^2=(1+1.5264)^2=6.3827$

$$Z_0=60\cosh^{-1}\left(\frac{S^2-a^2-a'^2}{2aa'}\right)\simeq 276\log_{10}\left(\frac{S}{\sqrt{aa'}}\right)=276\log_{10}\left[\frac{15.24}{1.9(0.635)}\right]=315.253$$

$$Z_t=jZ_0\tan\left(k\frac{l'}{2}\right)=j315.253\left[\frac{2\pi}{\lambda}(0.285\lambda)\right]=j57.06385$$

$$Z_{in}=\frac{2Z_t(1+\alpha)^2Z_a}{2Z_t+(1+\alpha)^2Z_a}=\frac{2(j57.06385)(6.3827)(25-j25)}{2(j57.06385)+6.3827(25-j25)}=155.2296\angle60.89^\circ$$

$Z_{in}=R_{in}+jX_{in}=75.505+j135.629$

b. $$X_{in}=\frac{1}{2\pi f C_{in}}\Rightarrow C_{in}=\frac{1}{2\pi f X_{in}}=\frac{1}{2\pi(15\times10^6)(135.629)}=78.231\times10^{-12}\text{f}$$

c. $C=2C_{in}=2(78.231\times10^{-12})=156.462\times10^{-12}\text{f}=156.462\text{ pf}$

9-22 According to the design procedure of Section 9.4.4

$$Z_2=(1+\alpha)^2\frac{Z_a}{2}\Rightarrow z_2=\frac{Z_2}{Z_0}\Rightarrow y_2=\frac{1}{z_2}=g_2+jb_2$$

$$z_g=\frac{Z_g}{Z_0}\Rightarrow y_g=\frac{1}{z_g}=g_g+jb_g$$

$$y_{in}=y_2+y_g=(g_2+jb_2)+(g_g+jb_g)=(g_2+g_g)+j(b_2+b_g)$$

$$z_{in}=\frac{1}{y_{in}}=\frac{1}{(g_2+g_g)+j(b_2+b_g)}\cdot\frac{(g_2+g_g)-j(b_2+b_g)}{(g_2+g_g)-j(b_2+b_g)}=\frac{(g_2+g_g)-j(b_2+b_g)}{(g_2+g_g)^2+(b_2+b_g)^2}$$

$z_{in}=r_{in}+jx_{in}\Rightarrow Z_{in}=z_{in}Z_0=(r_{in}+jx_{in})Z_0=R_{in}+jX_{in}$

$$R_{in}=r_{in}Z_0=\frac{(g_2+g_g)Z_0}{(g_2+g_g)^2+(b_2+b_g)^2}$$

$$X_{in}=x_{in}Z_0=\frac{(b_2+b_g)Z_0}{(g_2+g_g)^2+(b_2+b_g)^2}$$

(Continued)

9-22 (Cont'd) To find the length $\ell'/2$ which will perform the gamma match, we solve Rin for $b_g$. Thus

$$(g_2+g_g)Z_0 = R_{in}[(g_2+g_g)^2+(b_2+b_g)^2]$$

$$R_{in}(b_2+b_g)^2 = (g_2+g_g)Z_0 - R_{in}(g_2+g_g)^2$$

$$(b_2+b_g)^2 = [(g_g+g_2)Z_0 - R_{in}(g_2+g_g)^2]/R_{in}$$

$$b_2+b_g = \pm\sqrt{\frac{(g_2+g_g)Z_0 - R_{in}(g_2+g_g)^2}{R_{in}}}$$

$$b_g = -b_2 \pm \sqrt{\frac{(g_2+g_g)Z_0 - R_{in}(g_2+g_g)^2}{R_{in}}}$$

For Problem 9.22, $Z_a = (14+j3)$, $R_{in}=50$ and

$a = 0.9525/2 = 0.47625$ cm
$a' = 0.2052/2 = 0.1026$ cm
$S = 1.5316$ cm

$$u = a/a' = 0.47625/(0.1026) = 4.6418$$

$$\Rightarrow v = S/a' = 1.5316/(0.1026) = 14.92787$$

$$\alpha \simeq \frac{\ln v}{\ln v - \ln u} = \frac{\ln(14.92787)}{\ln(14.92787) - \ln(4.648)} = 2.314156$$

$$Z_g = jZ_0\tan(k\tfrac{\ell'}{2}) \Rightarrow Z_g = \frac{Z_g}{Z_0} = j\tan(k\tfrac{\ell'}{2}) \Rightarrow b_g = \frac{1}{Z_g} = -j\cot(k\tfrac{\ell'}{2}),\ g_g = 0$$

$$(1+\alpha)^2 = (1+2.31415)^2 = 10.9836$$

$$Z_0 \simeq 276\log_{10}\left(\frac{S}{\sqrt{aa'}}\right) = 276\log_{10}\left(\frac{1.5316}{0.47625(0.1026)}\right) = 232.02$$

$$Z_2 = \frac{Z_2}{Z_0} = \frac{(1+\alpha)^2 Z_a/2}{Z_0} = \frac{10.9836(14+j3)/2}{232.02} = 0.3389\angle 12.09^\circ = 0.331 + j0.071$$

$$y_2 = \frac{1}{Z_2} = \frac{1}{0.3389\angle 12.09^\circ} = 2.884 - j0.618 = g_2 + jb_2 \Rightarrow g_2 = 2.884,\ b_2 = -0.618$$

$$b_g = -\cot(k\tfrac{\ell'}{2}) = -b_2 \pm \sqrt{\frac{(g_2+g_g)Z_0 - R_{in}(g_2+g_g)^2}{R_{in}}}$$

$$b_g = -\cot(k\tfrac{\ell'}{2}) = -(-0.618) \pm \sqrt{\frac{2.884(232.02) - 50(2.884)^2}{50}} = 0.618 \pm 2.25$$

Using the minus (−) sign $(b_2+b_g = -2.25)$

$$\cot(k\tfrac{\ell'}{2}) = 1.632 \Rightarrow \tan(k\tfrac{\ell'}{2}) = 1/1.632 = 0.6127 \Rightarrow k\tfrac{\ell'}{2} = \tan^{-1}(0.6127)$$

$$\ell'/2 = \lambda/2\pi \cdot \tan^{-1}(0.6127) = \frac{\lambda}{2\pi}(0.5497) = 0.0875\lambda$$

Therefore

$$X_{in} = -\frac{(b_2+b_g)Z_0}{(g_2+g_g)^2+(b_2+b_g)^2} = \frac{-(-2.25)232.02}{(2.884)^2+(-2.25)^2} = 39.01 = \frac{1}{2\pi f C}$$

$$C = 1/(2\pi f X_{in}) = 1/(2\pi \times 145.4\times 10^6 \times 39.01) = 28.06\times 10^{-12} f$$

Using the procedure of Section 9.4.4, the Smith Chart Solution verifies the answers of the analytical method.

9-23 Using the procedure of the solution of Problem 9.22

$$Z_2 = 0.331 - j0.071 \Rightarrow y_2 = \frac{1}{Z_2} = 2.95\angle 12.09^\circ = 2.884 + j0.618 = g_2 + jb_2$$

Therefore $g_2 = 2.884$, $b_2 = 0.618$

also $g_g = 0$, $b_g = -\cot(k\frac{l'}{2})$

In addition

$$b_g = -\cot(k\tfrac{l'}{2}) = -b_2 \pm 2.25 = -0.618 \pm 2.25$$

Choosing the minus sign $(b_g + b_2) = -2.25$

$$\cot(k\tfrac{l'}{2}) = 2.868 \Rightarrow \tan(k\tfrac{l'}{2}) = 1/2.868 = 0.3487 \Rightarrow k\tfrac{l'}{2} = \tan^{-1}(0.3487)$$

$$\frac{l'}{2} = \frac{\lambda}{2\pi}\tan^{-1}(0.3487) = \frac{\lambda}{2\pi}(0.3355) = 0.0534\lambda$$

Therefore

$$X_{in} = -\frac{(b_2+b_g)Z_0}{(g_2+g_g)^2+(b_2+b_g)^2} = -\frac{(-2.25)(232.02)}{(2.884)^2+(2.25)^2} = 39.01 = \frac{1}{2\pi f C}$$

$$C = \frac{1}{2\pi f X_{in}} = \frac{1}{2\pi(145.4\times10^6)(39.01)} = 28.06\times10^{-12} f$$

Using the procedure of Section 9.4.4, the smith chart solution verifies the answers of the above analytical method.

9-24 $Z_a = 34 + j17$, $R_{in} = 50$

$d' = d = 0.9525$ cm $\Rightarrow a' = a = 0.9525/2 = 0.47625$ cm

$S = 3.1496$ cm $\Rightarrow S'/2 = 3.1496/2 = 1.5748$ cm

$$Z_0 = 120\cdot\ln\left[\frac{S/2+\sqrt{(S/2)^2-a^2}}{a}\right] = 120\cdot\ln\left[\frac{1.5748+\sqrt{(1.5748)^2-(0.47625)^2}}{0.47625}\right] = 223.85$$

$u = a/a' = 1$

$v = S/a' = 1.5748/0.47625 = 3.3067$

$$\alpha \simeq \frac{\ln v}{\ln v - \ln u} = \frac{\ln(3.3067)}{\ln(3.3067)-\ln(1)} = 1 \Rightarrow (1+\alpha)^2 = (1+1)^2 = 4$$

$$Z_{12} = (1+\alpha)^2\frac{Z_a}{2} = 4\left(\frac{34+j17}{2}\right) = 76.026\angle 26.565^\circ \Rightarrow z_2 = \frac{Z_{12}}{Z_0} = \frac{76.026\angle 26.565^\circ}{223.85}$$

$$z_2 = 0.3396\angle 26.565^\circ \Rightarrow y_2 = \frac{1}{z_2} = \frac{1}{0.3396\angle 26.565^\circ} = 2.6335 - j1.317 = g_2 + jb_2$$

Therefore $g_2 = 2.6335$, $b_2 = -1.317$

$$Z_g = jZ_0\tan(k\tfrac{l'}{2}) \Rightarrow z_g = \frac{Z_g}{Z_0} = j\tan(k\tfrac{l'}{2}) \Rightarrow y_g = \frac{1}{z_g} = -j\cot(k\tfrac{l'}{2}) = g_g + jb_g$$

Thus $g_g = 0$, $b_g = -\cot(k\frac{l'}{2})$

Using the equations derived in the first part of the solution of Problem 9.22, we have that

(Continued)

9-24 (Cont'd)

$$b_g = -b_2 \pm \sqrt{\frac{(g_2+g_g)Z_0 - R_{in}(g_2+g_g)^2}{R_{in}}}$$

$$= -(-1.317) \pm \sqrt{\frac{(2.6335)(223.85) - 50(2.6335)^2}{50}}$$

$$b_g = -\cot\left(k\frac{l'}{2}\right) = 1.317 \pm 2.2034$$

Choosing the minus sign $[(b_2+b_g) = -2.2034]$

$$\cot\left(k\frac{l'}{2}\right) = 0.8864 \Rightarrow \tan\left(k\frac{l'}{2}\right) = 1/0.8864 = 1.128 \Rightarrow k\frac{l'}{2} = \tan^{-1}(1.128)$$

$$\frac{l'}{2} = \frac{\lambda}{2\pi}\tan^{-1}(1.128) = \frac{\lambda}{2\pi}(0.8455) = 0.1346\lambda$$

Therefore

$$X_{in} = -\frac{(b_2+b_g)Z_0}{(g_2+g_g)^2+(b_2+b_g)^2} = -\frac{(-2.2034)(223.85)}{(2.6335)^2+(-2.2034)^2} = 41.834 = \frac{1}{2\pi f C}$$

$$C = \frac{1}{2\pi f X_{in}} = \frac{1}{2\pi(145.4\times10^6)41.834} = 26.165\times10^{-12} f$$

Using the procedure of Section 9.4.4, the smith chart solution verifies the answers of the above analytical method

9-25. a. $Z_a = 73 + j42.5$

$\alpha = 1$

$Z_a/2 = 36.5 + j21.25$

$$Z_d = (1+\alpha)^2 Z_a/2 = 4\left(\frac{73+j42.5}{2}\right) = 2(73+j42.5)$$

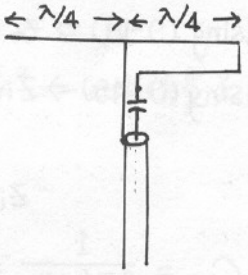
$$Z_g = j Z_0 \tan\left(k\frac{l'}{2}\right) = j Z_0 \cdot \tan\left(\frac{2\pi}{\lambda}\frac{\lambda}{4}\right) = j Z_0 \tan\left(\frac{\pi}{2}\right) = \infty$$

$$Z_{in} = Z_d = 2(73+j42.5) = 146 + j85$$

b. $$X_C = 85 = \frac{1}{2\pi f C}$$

$$C = \frac{1}{2\pi f X_C} = \frac{1}{2\pi(425\times10^6)(85)} = 4.406\times10^{-12}$$

$$C = 4.406\times10^{-12} = 4.406\ pF$$




c. $$\Gamma = \frac{Z_{in} - Z_c}{Z_{in} + Z_c} = \frac{146-78}{146+78} = 0.3036$$

$$VSWR = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.3036}{1-0.3036} = 1.872$$

9-26. $Z_2 = (1+\alpha)^2 \frac{Z_a}{2} = 3^2 \cdot \frac{67}{2} = 301.5$

a. $Z_g = j Z_0 \tan(k\frac{\ell'}{2}) = j Z_0 \tan(\frac{2\pi}{\lambda} \cdot 0.1\lambda) = j Z_0 \tan(36°) = j250(0.7265)$

$Z_g = j181.6356$

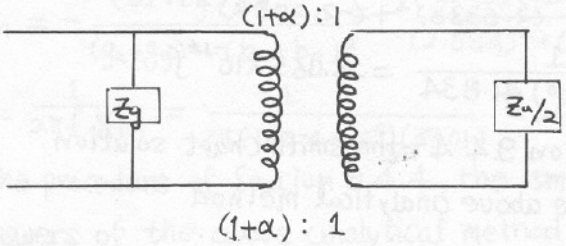
$Z_{in}' = \frac{Z_g Z_2}{Z_g + Z_2} = \frac{j(181.6356)(301.5)}{301.5 + j181.6356} = 155.583 \angle 58.934°$

$Z_{in}' = 155.583 \angle 58.934° = 80.285 + j133.268$

b. $X_C = \frac{1}{2\pi f C} = 133.268$

$C = \frac{1}{2\pi(133.268)\,3\times10^8} = 3.98 \times 10^{-4} \times 10^{-8} = 3.980 \times 10^{-12}$ farads

$C = 3.980 \times 10^{-12}$ farads $= 3.98$ pf




9-27. a. From the solution of Problem 9.22

$Z_0 = 232.02$, $Z_a = 14 + j3$, $(1+\alpha)^2 = 10.9836$, $\frac{\ell'}{2} = 0.0875\lambda$, $\tan(k\frac{\ell'}{2}) = 0.6127$

Using (9-18) $\Rightarrow Z_t = j Z_0 \tan(k\frac{\ell'}{2}) = j232.02(0.6127) = j142.158$

Using (9-49) $\Rightarrow Z_{in} = \frac{2Z_t(1+\alpha)^2 Z_a}{2Z_t + (1+\alpha)^2 Z_a} = \frac{2(j142.158)(10.9836)(14+j3)}{2(j142.158) + 10.9836(14+j3)}$

$Z_{in} = 100 + j77.9865$

$C_{in} = 2\pi f X_{in} = \frac{1}{2\pi(145.4\times10^6)(77.9865)} = 14.036 \times 10^{-12} f \Rightarrow C = 2C_{in} = 28.072 \times 10^{-12} f$

b. From the solution of Problem 9.23

$Z_0 = 232.02$, $Z_a = 14 - j3$, $(1+\alpha)^2 = 10.9836$, $\frac{\ell'}{2} = 0.0534\lambda$, $\tan(k\frac{\ell'}{2}) = 0.3487$

Using (9-18) $\Rightarrow Z_t = j Z_0 \tan(k\frac{\ell'}{2}) = j232.02(0.3487) = j80.9054$

Using (9-49) $\Rightarrow Z_{in} = \frac{2Z_t(1+\alpha)^2 Z_a}{2Z_t + (1+\alpha)^2 Z_a} = \frac{2(j80.9054)(10.9836)(14-j3)}{2(j80.9054) + 10.9836(14-j3)}$

$Z_{in} = 100.02 + j78.002$

$C_{in} = \frac{1}{2\pi f X_{in}} = \frac{1}{2\pi(145.4\times10^6)(78.002)} = 14.03 \times 10^{-12} f$

$C = 2C_{in} = 28.06 \times 10^{-12} f = 28.06$ pF

9-28. By comparing (9-49) with (9-54), it is apparent that the second term of (9-54) is equal to one-half of (9-49). Therefore the lengths of the T-match rods that will match to a given input resistance can be obtained by following the procedure of the gamma match of Problem 9.22 but with an input resistance of one-half that desired for the T-match.

$d' = 0.1026 \Rightarrow a' = d'/2 = 0.0513\text{cm}$

$d = 0.9525 \Rightarrow a = d/2 = 0.47625\text{ cm}$

$S = 0.7658 \Rightarrow S/2 = 0.3829\text{ cm}$

$U = a/a' = 0.47625/0.0513 = 9.2836$

$V = S/a' = 0.7658/0.0513 = 14.9278$

$$\alpha = \frac{\ln V}{\ln V - \ln U} = \frac{\ln(14.9278)}{\ln(14.9278) - \ln(9.2836)} = 5.6912 \Rightarrow (1+\alpha)^2 = 44.177$$

$$Z_0 \simeq 276 \log_{10}\left(\frac{S}{\sqrt{aa'}}\right) = 276 \log_{10}\left(\frac{0.7658}{\sqrt{(0.47625)(0.0513)}}\right) = 190.478$$

To find the lengths of the T-match rods so that the input impedance is 300 ohms, use the gamma match procedure outlined in the solution of Problem 9.22 but for a 150 ohm input resistance.

a. For Prob. 9.22 $\Rightarrow Z_a = 14 + j3$, $f = 145.4\text{ MHz}$

$$Z_2 = (1+\alpha)^2 \frac{Z_a}{2} = 44.177\left(\frac{14+j3}{2}\right) = 320.51\angle 12.09^\circ$$

$$z_2 = \frac{Z_2}{Z_0} = \frac{320.51\angle 12.09^\circ}{190.478} = 1.6827\angle 12.09^\circ = 1.6453 + j0.3524$$

$$y_2 = \frac{1}{z_2} = \frac{1}{1.6827\angle 12.09^\circ} = 0.59428\angle -12.09^\circ = 0.5811 - j0.12447 = g_2 + jb_2$$

Therefore $\Rightarrow g_2 = 0.5811$, $b_2 = -0.12447$

$$z_g = j\tan\left(k\frac{l'}{2}\right) \Rightarrow y_g = -j\cot\left(k\frac{l'}{2}\right) = g_g + jb_g \Rightarrow b_g = -\cot\left(k\frac{l'}{2}\right),\ g_g = 0$$

$$b_g = -b_2 \pm \sqrt{\frac{(g_2+g_g)Z_0 - R_{in}(g_2+g_g)^2}{R_{in}}}$$

$$b_g = -(-0.12447) \pm \sqrt{\frac{0.5811(190.478) - 150(0.5811)^2}{150}} = -\cot\left(k\frac{l'}{2}\right)$$

$$b_g = 0.12447 \pm 0.63264 = -\cot\left(k\frac{l'}{2}\right)$$

Choosing the minus sign $(b_2 + b_g = -0.63264)$

$$\cot\left(k\frac{l'}{2}\right) = 0.50817 \Rightarrow \tan\left(k\frac{l'}{2}\right) = 1.96784 \Rightarrow k\frac{l'}{2} = \tan^{-1}(1.96784) = 1.1006$$

$$\frac{l'}{2} = \frac{\lambda}{2\pi}(1.1006) = 0.1752\lambda$$

(Continued)

9-28. (Cont'd)

Using (9-18) $\Rightarrow Z_t = jZ_0 \tan(k\frac{\ell'}{2}) = j190.478(1.96784) = j374.83$

Using (9-49), the input impedance of the T-match is equal to

$$Z_{in} = \frac{2Z_t(1+\alpha)^2 Z_a}{2Z_t + (1+\alpha)^2 Z_a} = \frac{2(j374.83)44.77(14+j3)}{2(j374.83)+44.77(14+j3)} = 443.4579\angle 47.4286^\circ$$

$$Z_{in} = 300.003 + j326.5778$$

$$C_{in} = \frac{1}{2\pi f X_{in}} = \frac{1}{2\pi(145.4\times10^6)326.5778} = 3.352\times10^{-12} f$$

$$C = 2C_{in} = 2(3.352\times10^{-12}) = 6.704\times10^{-12} f$$

b. For Problem 9.23 $\Rightarrow Z_a = 14-j3$, $f = 145.4$ MHz

$$Z_{12} = (1+\alpha)^2\frac{Z_a}{2} = 320.51\angle -12.09^\circ,\quad Z_2 = \frac{Z_{12}}{Z_0} = 1.6821\angle -12.09^\circ$$

$$y_2 = \frac{1}{Z_2} = 0.59428\angle 12.09^\circ = 0.5811 + j0.12447 = g_2 + jb_2$$

Therefore $\Rightarrow g_2 = 0.5811$, $b_2 = 0.12447$

$$Z_g = j\tan(k\tfrac{\ell'}{2}) \Rightarrow y_g = -j\cot(k\tfrac{\ell'}{2}) = g_g + jb_g \Rightarrow g_g = 0,\ b_g = -\cot(k\tfrac{\ell'}{2})$$

$$b_g = -b_2 \pm \sqrt{\frac{(g_2+g_g)Z_0 - R_{in}(g_2+g_g)^2}{R_{in}}} = -0.12447 \pm \sqrt{\frac{0.5811(190.478)-150(0.5811)^2}{150}}$$

$$b_g = -\cot(k\tfrac{\ell'}{2}) = -0.12447 \pm 0.63264$$

Choosing the minus sign $(b_2 + b_g = -0.63264)$

$$\cot(k\tfrac{\ell'}{2}) = 0.75711 \Rightarrow \tan(k\tfrac{\ell'}{2}) = 1.32081 \Rightarrow k\tfrac{\ell'}{2} = \tan^{-1}(1.32081)$$

$$\frac{\ell'}{2} = \frac{\lambda}{2\pi}\tan^{-1}(1.32081) = \frac{\lambda}{2\pi}(0.9227597) = 0.14686\lambda$$

Using (9-18) $\Rightarrow Z_t = jZ_0\tan(k\tfrac{\ell'}{2}) = j190.478(1.32081) = j251.585$

Using (9-49), the input impedance of the T-match is equal to

$$Z_{in} = \frac{2Z_t(1+\alpha)^2 Z_a}{2Z_t + (1+\alpha)^2 Z_a} = \frac{2(j251.585)44.77(14-j3)}{2(j251.585)+44.77(14-j3)} = 443.4993\angle 47.4332^\circ$$

$$Z_{in} = 300.005 + j326.6324$$

$$C_{in} = \frac{1}{2\pi f X_{in}} = \frac{1}{2\pi(145.4\times10^6)326.6324} = 3.351\times10^{-12} f$$

$$C = 2C_{in} = 6.702\times10^{-12} f$$

10-1. $\underline{I} = \hat{a}_z I_0 e^{-jk_z z'}$

a. Using (4-57a) – (4-57c)

$E_r \simeq E_\phi \simeq H_r \simeq H_\theta = 0$

$$E_\theta = \int_0^\ell dE_\theta = \int_0^\ell j\eta \frac{kI_0 e^{-jk_z z'} e^{-jkR}}{4\pi R} \sin\theta dz'$$

Since for far-field observations

$R \simeq r - z'\cos\theta$ for phase terms

$R \simeq r$ for amplitude terms

$$E_\theta \simeq j\eta \frac{kI_0 e^{-jkr}}{4\pi r} \sin\theta \int_0^\ell e^{+j(kz'\cos\theta - k_z z')} dz'$$

$$\simeq j\eta \frac{kI_0 e^{-jkr}}{4\pi r} \sin\theta \int_0^\ell \frac{e^{j(k\cos\theta - k_z)z'}}{j(k\cos\theta - k_z)} d[j(k\cos\theta - k_z)z']$$

$$\simeq j\eta \frac{kI_0 e^{-jkr}}{4\pi r} \sin\theta \left[ \frac{e^{j(k\cos\theta - k_z)z'}}{j(k\cos\theta - k_z)} \right]_0^\ell$$

$$E_\theta \simeq j\eta \frac{kI_0 e^{-jkr}}{4\pi r} \cdot e^{-j\frac{k\ell}{2}(K - \cos\theta)} \cdot \sin\theta \cdot \frac{\sin\left[\frac{k\ell}{2}(\cos\theta - K)\right]}{\frac{k\ell}{2}(\cos\theta - K)}$$

where $K = \frac{k_z}{k}$

$$H_\phi \simeq \frac{E_\theta}{\eta}$$

b. $\underline{W}_{ave} = \frac{1}{2} Re(\underline{E} \times \underline{H}^*) = \frac{1}{2} Re[\hat{a}_\theta E_\theta \times \hat{a}_\phi H_\phi^*]$

$$= \frac{1}{2} Re\left(\hat{a}_\theta E_\theta \times \hat{a}_\phi \frac{E_\theta^*}{\eta}\right) = \frac{\hat{a}_r}{2\eta} |E_\theta|^2$$

which for $K = k_z/k = 1$ reduces to

$$\underline{W}_{ave} = \hat{a}_r \eta \frac{|I_0|^2}{8\pi^2 r^2} \frac{\sin^2\theta}{(\cos\theta - 1)^2} \sin^2\left[\frac{k\ell}{2}(\cos\theta - 1)\right]$$

Using the trigonometric identify (from Appendix VI)

$$\cot\left(\frac{x}{2}\right) = \frac{\sin x}{1 - \cos x}$$

it reduces to

$$\underline{W}_{ave} = \hat{a}_r \eta \frac{|I_0|^2}{8\pi^2 r^2} \cot^2\left(\frac{\theta}{2}\right) \sin^2\left[\frac{k\ell}{2}(\cos\theta - 1)\right]$$

(Continued)

10-1 (Cont'd)

C. $P_{rad} = \oint_S \underline{W}_{ave} \cdot d\underline{s} = \int_0^{2\pi}\int_0^{\pi} \hat{a}_r W_r \cdot \hat{a}_r r^2 \sin\theta \, d\theta \, d\phi$

$$P_{rad} = \eta \frac{|I_0|^2 (2\pi)}{8\pi^2} \int_0^{\pi} \cot^2\left(\frac{\theta}{2}\right) \sin^2\left[\frac{kl}{2}(\cos\theta - 1)\right] \sin\theta \, d\theta$$

which reduces to

$$P_{rad} = \eta \frac{|I_0|^2}{4\pi}\left[1.415 + \ln\left(\frac{kl}{\pi}\right) - C_i(2kl) + \frac{\sin(2kl)}{2kl}\right]$$

10-2. For a long wire antenna, the $\sin\theta$ variations in (10-2b) are much slower than those of the $\frac{\sin\left[\frac{kl}{2}(\cos\theta - K)\right]}{\frac{kl}{2}(\cos\theta - K)}$ function.

Therefore near the axis

$$E_\theta = j\eta \frac{kI_0 l e^{-jkr}}{4\pi r} \cdot e^{-j\frac{kl}{2}(K - \cos\theta)} \cdot \sin\theta \, \frac{\sin\left[\frac{kl}{2}(\cos\theta - K)\right]}{\frac{kl}{2}(\cos\theta - K)}$$

$$\simeq j\eta \frac{kl I_0 e^{-jkr}}{4\pi r} e^{j\frac{kl}{2}(K - \cos\theta)} \cdot \frac{\sin\left[\frac{kl}{2}(\cos\theta - K)\right]}{\frac{kl}{2}(\cos\theta - K)}$$

The first maximum occurs when

$$\frac{kl}{2}(\cos\theta_m - K) = \frac{kl}{2}\left(\cos\theta_m - \frac{k_z}{k}\right) \simeq 0 \Rightarrow K = \frac{v_p}{v_0} = \frac{k_z}{k} \simeq \cos\theta_m$$

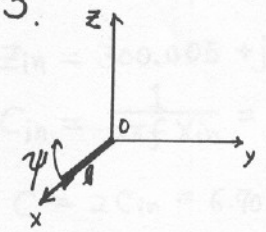
a. $\frac{v_p}{v_0} = \cos\theta_m\Big|_{\theta_m = 10^\circ} = \cos(10^\circ) = 0.9848 \Rightarrow v_p = 0.9848\, v_0$

$v_p = 0.9848\,(3\times10^8)$, $v_p = 2.9544 \times 10^8$ m/sec.

b. $\frac{v_p}{v_0} = \cos\theta_m\Big|_{\theta_m = 20^\circ} = \cos(20^\circ) = 0.93969 \Rightarrow v_p = 0.93969\, v_0$

$v_p = 0.93969\,(3\times10^8)$, $v_p = 2.8191 \times 10^8$ m/sec

10-3.




$$\underline{A} = \hat{a}_x \frac{\mu I_0}{4\pi} \int_0^l e^{-jkx'} \frac{e^{-jkR}}{R} dx'$$

far field: $\begin{cases} R = r - x'\cos\psi & \text{phase} \\ R = r & \text{Amplitude} \end{cases}$

(Continued)

10-3. (Cont'd)

$$A_x = \frac{\mu I_0 e^{-jkr}}{4\pi r}\int_0^{\ell} e^{-jk(1-\cos\psi)x'}dx'$$

$$= \frac{\mu I_0 e^{-jkr}}{4\pi r}\left[\frac{e^{-jk(1-\cos\psi)x'}}{-jk(1-\cos\psi)}\right]_0^{\ell}$$

$$A_x = \frac{\mu I_0 \ell e^{-jkr}}{4\pi r}\left[\frac{e^{-jk(1-\cos\psi)\ell}}{-jk(1-\cos\psi)} + \frac{1}{jk(1-\cos\psi)}\right]$$

$$= \frac{\mu I_0 \ell e^{-jkr}}{4\pi r} e^{-jX}\cdot\frac{\sin X}{X}, \text{ where } X = \frac{k\ell}{2}(1-\cos\psi)$$

Convert to spherical coordinates

$$\left.\begin{aligned} A_r &= A_x \sin\theta\cos\phi \\ A_\theta &= A_x \cos\theta\cos\phi \\ A_\phi &= -A_x \sin\phi \end{aligned}\right\} \text{ Far-field} \Rightarrow \left\{\begin{aligned} E_\theta &= -j\omega A_\theta \\ E_\phi &= -j\omega A_\phi \\ E_r &= 0 \end{aligned}\right.$$

$$E_\theta = -j\frac{\omega\mu I_0 \ell e^{-jkr}}{4\pi r}\cdot e^{-jX}\cdot\frac{\sin X}{X}\cos\theta\cos\phi \qquad , \cos\psi = |\hat{a}_x\cdot\hat{a}_r|$$

$$, \cos\psi = \sin\theta\cdot\cos\phi$$

$$E_\phi = +j\frac{\omega\mu I_0 \ell e^{-jkr}}{4\pi r}\cdot e^{-jX}\cdot\frac{\sin X}{X}\cdot\sin\phi$$

$$H_\phi = \frac{E_\theta}{\eta}, \quad H_\theta = -\frac{E_\phi}{\eta}, \quad X = \frac{k\ell}{2}(1-\cos\psi), \; X = \frac{k\ell}{2}(1-\sin\theta\cos\phi)$$

10-4. In the far-zone;

a. $$\underline{A}(\underline{r}) \simeq \hat{a}_z\frac{\mu e^{-jkr}}{4\pi r}\int_0^{\ell} I(z')e^{jkz'\cos\theta}dz'$$

$$= \hat{a}_z\frac{\mu I_0 e^{-jkr}}{4\pi r}[1+R]^{-1}\cdot\left\{\int_0^{\ell} e^{-jkz'}\cdot e^{jkz'\cos\theta}dz' + R\int_0^{\ell} e^{jkz'}\cdot e^{jkz'\cos\theta}dz'\right\}$$

$$\leftarrow \int_0^{\ell} e^{-jkz'}e^{jkz'\cos\theta}dz' = \int_0^{\ell} e^{jkz'(1-\cos\theta)}dz' \underset{\text{let } \xi = z'-\frac{\ell}{2}}{=\!=\!=} e^{j\frac{k\ell}{2}(1-\cos\theta)}\cdot\int_{-\ell/2}^{\ell/2} e^{jk\xi(1-\cos\theta)}d\xi$$

$$= \ell e^{jA}\cdot\frac{\sin(A)}{A}, \quad A = \frac{k\ell}{2}(1-\cos\theta)$$

(Continued)

10-4. (Cont'd)

$$\leftarrow \int_0^{l} e^{jkz'} e^{jkz'\cos\theta}\, dz' = \int_0^{l} e^{jkz'(1+\cos\theta)}\, dz' \quad , \text{ let } \xi = z' - l/2$$

$$= e^{j\frac{kl}{2}(1+\cos\theta)} \int_{-l/2}^{l/2} e^{jk\xi(1+\cos\theta)}\, d\xi$$

$$= l\, e^{jB} \cdot \frac{\sin(B)}{B} \quad , B = \frac{kl}{2}(1+\cos\theta)$$

$$\therefore A_z = \frac{\mu I_0 l}{4\pi} \cdot \frac{e^{jkr}}{r} \left\{ \frac{e^{jA}\cdot \frac{\sin(A)}{A} + R e^{jB}\cdot \frac{\sin(B)}{B}}{1+R} \right\}$$

$$A_\theta = -A_z \cdot \cos\theta \simeq -\frac{\mu I_0 l\, e^{jkr}}{4\pi r} \cdot \left\{ \frac{e^{jA}\cdot \sin(A)/A + R e^{jB}\cdot \sin(B)/B}{1+R} \right\} \sin\theta$$

$$A_\phi = 0 \ , \ E_r \simeq 0 \quad E_\phi \simeq 0$$

$$E_\theta \simeq -j\omega A_\theta = j\frac{2kI_0 l\, e^{jkr}}{4\pi r} \left\{ \frac{e^{jA}\cdot \sin(A)/A + R e^{jB}\cdot \sin(B)/B}{1+R} \right\} \cdot \sin\theta$$

b. $$W_{ave} \simeq \frac{|E_\theta|^2}{2\eta} \simeq \frac{\eta (kl)^2}{32\pi^2 r^2} |I_0|^2 \left| \left\{ \frac{e^{jA}\cdot \sin(A)/A + R e^{jB}\sin(B)/B}{1+R} \right\} \right|^2 \cdot \sin^2\theta$$

$$U(\theta,\phi) = \lim_{r\to\infty} r^2 W_{ave} = \frac{\eta (kl)^2}{32\pi^2} |I_0|^2 \cdot \left| \{ \downarrow \} \right|^2 \cdot \sin^2\theta$$

$$= \frac{\eta}{8} \left(\frac{l}{\lambda}\right)^2 \cdot |I_0|^2 \cdot \left| \left\{ \frac{e^{jA}\cdot \sin(A)/A + R e^{jB}\cdot \sin(B)/B}{1+R} \right\} \right|^2 \cdot \sin^2\theta$$

$\theta = \frac{\pi}{2}$

$A = \frac{kl}{2}$

$B = \frac{kl}{2}$

$\sin\theta = 1.$

$$U(\theta=\tfrac{\pi}{2},\phi) = \frac{\eta}{8}\left(\frac{l}{\lambda}\right)^2 \cdot |I_0|^2 \cdot \frac{\sin^2\left(\frac{kl}{2}\right)}{\left(\frac{kl}{2}\right)^2} \left| \frac{e^{-j\frac{kl}{2}} + R e^{+j\frac{kl}{2}}}{1+R} \right|^2$$

10-5. Using (10-7), the first maximum occurs exactly when $2m+1 = 0.742$ and approximately when $2m+1 \simeq 1$. Thus

a. Exact; $\theta_m = \cos^{-1}\left[1 \pm \frac{\lambda}{2\ell}(0.742)\right] = 10° \Rightarrow \cos(10°) = 1 \pm \frac{\lambda}{2\ell}(0.742)$

$$\ell = \frac{0.742\lambda}{2(0.01519)} = 24.42\lambda = \ell_{exact}$$

Approx; $\theta_m = \cos^{-1}\left[1 \pm \frac{\lambda}{2\ell}(1)\right] = 10° \Rightarrow \cos(10°) = 1 \pm \frac{\lambda}{2\ell}(1)$

$$\ell = \frac{\lambda}{2(0.01519)} = 32.9115\lambda = \ell_{approx}.$$

b. Using (10-7), $\Rightarrow \theta_m = \cos^{-1}\left[1 - \frac{\lambda}{2\ell}(2m+1)\right]$ with the minus sign and with $\ell_{exact} = 24.42\lambda$ and $(2m+1) = 0.742, 2.93, 4.96, 6.97, 8.99,$ 11 and 13.

$\theta_{m0} = 10.00°$, $\theta_{m1} = 19.95°$, $\theta_{m2} = 26.05°$, $\theta_{m3} = 30.99°$, $\theta_{m4} = 35.32°$

$\theta_{m5} = 39.22°$, $\theta_{m6} = 42.79°$

with $\ell_{approx} \simeq 32.9115\lambda$ and $2m+1 = 1, 3, 5, 7, 9, 11, 13$

$\theta_{m0} \simeq 10.00°$, $\theta_{m1} \simeq 17.36°$, $\theta_{m2} \simeq 22.48°$, $\theta_{m3} \simeq 26.66°$, $\theta_{m4} \simeq 30.31°$

$\theta_{m5} \simeq 33.60°$, $\theta_{m6} \simeq 36.63°$

c. Using (10-10) $\Rightarrow \theta_n = \cos^{-1}(1 - n\frac{\lambda}{L})$, $n = 1, 2, 3, 4 \cdots$ with

| | $\ell_{exact} = 24.42\lambda$ | $\ell_{approx} \simeq 32.9115\lambda$ |
|---|---|---|
| $\theta_{n0}$ = | 16.45° | 14.16° |
| $\theta_{n1}$ = | 23.35° | 20.08° |
| $\theta_{n2}$ = | 28.70° | 24.65° |
| $\theta_{n3}$ = | 33.26° | 28.54° |
| $\theta_{n4}$ = | 37.32° | 32.00° |
| $\theta_{n5}$ = | 41.04° | 35.15° |

d. Using (10-12) $\Rightarrow R_r = 60\left[1.415 + \ln\left(\frac{k\ell}{\pi}\right) - C_i(2k\ell) + \frac{\sin(2k\ell)}{2k\ell}\right]$

$\ell = \ell_{exact} = 24.42\lambda$;

$k\ell/\pi = 48.84 \Rightarrow \ln(k\ell/\pi) = \ln(48.84) = 3.88855$

$2k\ell = 306.87077 \Rightarrow C_i(2k\ell) = C_i(306.87077) \simeq 0$ from Fig III-1.

$\sin(2k\ell)/2k\ell = \sin(306.87077)/306.87077 = -0.00275$

$R_r = 60[1.415 + 3.88855 - 0.00275] = 318.05$

(Continued)

10-5. (Cont'd)

$l = l_{approx} = 32.9115\lambda$ ;

$\frac{kl}{\pi} = 65.823 \Rightarrow \ln(kl/\pi) = \ln(65.823) = 4.186969$

$2kl = 413.5781 \Rightarrow C_i(2kl) = C_i(413.5781) \simeq 0$ from Fig III-1.

$\sin(2kl)/2kl = \sin(413.5781)/413.5781 = -0.002168$

$R_r = 60(1.415 + 4.186969 - 0.002168) = 335.988$

e. Using (10-13) $\Rightarrow D_o = \dfrac{2\cot^2\left[\frac{1}{2}\cos^{-1}\left(1 - \frac{0.371\lambda}{l}\right)\right]}{1.415 + \ln\left(\frac{2l}{\lambda}\right) - C_i(2kl) + \frac{\sin(2kl)}{2kl}}$

$l = l_{exact} = 24.42\lambda$

$\cot\left[\frac{1}{2}\cos^{-1}\left(1 - \frac{0.371}{24.42}\right)\right] = 11.42997$

$D_o = \dfrac{2(11.42997)^2}{5.3007986} = 49.29227 = 16.928$ dB

$l = l_{approx} = 32.9115\lambda$

$\cot\left[\frac{1}{2}\cos^{-1}\left(1 - \frac{0.371}{32.9115}\right)\right] = 13.282337.$

$D_o = \dfrac{2(13.282337)^2}{5.5998} = 63.00957 = 17.994$ dB

10-6. a. Exact;

$\theta_o = \cos^{-1}\left[1 \pm \frac{\lambda}{2l}(0.742)\right]$, $\cos\theta_o = 1 \pm \frac{\lambda}{2l}(0.742)$, $0.9063 = 1 \pm \frac{\lambda}{2l}(0.742)$

$-0.0937 = \pm\frac{\lambda}{2l}(0.742)$, $l = \dfrac{0.742\lambda}{2(0.0937)} = 3.96\lambda$

Approx;

$\theta_m \simeq \cos^{-1}\left[1 \pm \frac{\lambda}{2l}(2m+1)\right]$, $m = 0, 1, 2, \cdots$

$\theta_o \simeq \cos^{-1}\left[1 \pm \frac{\lambda}{2l}(1)\right] \Rightarrow \cos\theta_o \simeq 1 \pm \frac{\lambda}{2l}$

$\cos(25°) \simeq 1 + \frac{\lambda}{2l} = 0.9063$, $\pm\frac{\lambda}{2l} \simeq 0.9063 - 1 = 0.0937$

$l \simeq \dfrac{\lambda}{2(0.0937)} = 5.3366\lambda$

b. $kl = \frac{2\pi}{\lambda}(3.96\lambda) = 2\pi(3.96) = 24.88$, $2kl = 49.76$

$\sin(2kl)/2kl = -0.00999$, $\ln(kl/\pi) = \ln(7.42) = 2.0093$

$\ln(2l/\lambda) = \ln(2\cdot(3.71)\lambda/\lambda) = 2.0693$

$C_i(2kl) = C_i(49.76) \simeq 0$

$R_r = 60\left[1.415 + \ln\left(\frac{kl}{\pi}\right) - C_i(2kl) + \frac{\sin(2kl)}{2kl}\right] = 60[1.415 + 2.0693 - 0.00999] = 208.46$

(Continued)

10-6 (Cont'd)

c. $D_o = \dfrac{2\cdot\cot^2[\frac{1}{2}\cos^{-1}(1-0.371\lambda/\ell)]}{1.415+\ln(2\ell/\lambda)-C_i(2k\ell)+\frac{\sin(2k\ell)}{2k\ell}} = \dfrac{2\cot^2[\frac{1}{2}\cos^{-1}(1-\frac{0.371}{3.96})]}{1.415+2.0693-0-0.00999}$

$= \dfrac{40.6954}{3.47431} = 11.7132 = 10.686\,dB$

10-7. $D_o = \dfrac{2\cot^2[\frac{1}{2}\cos^{-1}(1-\frac{0.371\lambda}{\ell})]}{1.415+\ln(2\ell/\lambda)-C_i(2k\ell)+\sin(2k\ell)/2k\ell}$

a. $\ell = 2\lambda$;

$\cot[\frac{1}{2}\cos^{-1}(1-\frac{0.371}{2})] = 3.127566$

$\ln(2\ell/\lambda) = \ln(4) = 1.386294$

$C_i(2k\ell) = C_i(8\pi) \simeq 0$ from Fig III.1 (Appendix III)

$\sin(2k\ell)/(2k\ell) = \sin(8\pi)/(8\pi) \simeq 0$

$D_o = \dfrac{2(3.127566)^2}{1.415+1.386294} = 6.984 = 8.44\,dB$

b. $\ell = 3\lambda$;

$\cot[\frac{1}{2}\cos^{-1}(1-\frac{0.371}{3})] = 3.89519$

$\ln(2\ell/\lambda) = \ln(6) = 1.791759$

$C_i(2k\ell) = C_i(12\pi) \simeq 0$ from. Fig III-1 (Appendix III)

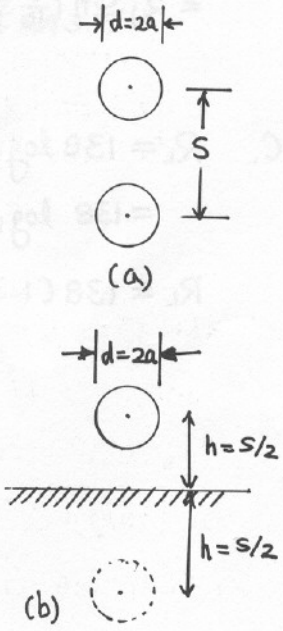
$\sin(2k\ell)/(2k\ell) = \sin(12\pi)/(12\pi) \simeq 0$

$D_o = \dfrac{2(3.89519)^2}{1.415+1.791759} = 9.4628 = 9.76\,dB$

10-8. a. The characteristic impedance of a lossless transmission line is given by $Z_o = \sqrt{\frac{L}{C}}$

where L is the inductance per unit length (h/m) and C is the capacitance per unit length (f/m) of the line. For a two-wire transmission Line, as shown in Fig. (a), the characteristic impedance is given by (9-19) or

$Z_o = 120\cosh^{-1}(\frac{S/2}{a}) = 120.\ln\left[\dfrac{S/2+\sqrt{(S/2)^2-a^2}}{a}\right]$




(Continued)

10-8 (Cont'd) which for $S \gg a$ reduces to

$$Z_a \overset{S\gg a}{\simeq} 120 \ln\left(\frac{S}{a}\right) = 120(2.3026)\log_{10}\left(\frac{S}{a}\right)$$

$$Z_0 \simeq 276.31 \log_{10}\left(\frac{S}{a}\right)$$

For a one wire above a ground plane transmission line, its characteristic impedance is equal to [see Fig. (b)]

$$Z = \frac{1}{2}Z_0 \simeq \frac{1}{2}(276.31)\log_{10}\left(\frac{S}{a}\right) = 138.155\log_{10}\left(\frac{S}{a}\right)$$

For $S/2 = h$ and $a = d/2$, it can be written as

$$Z \simeq 138.155\log_{10}\left(4\frac{h}{d}\right)$$, where h is the height of the wire, of diameter d, above the ground plane.

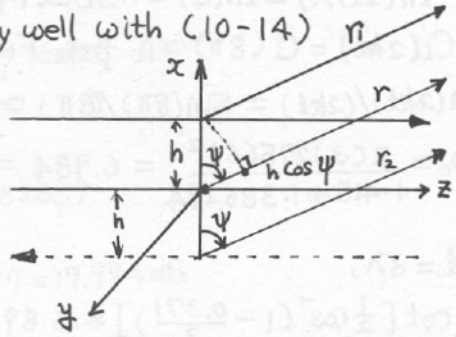
b. The expression above compares very well with (10-14)

10-9. a. $AF = e^{+jkh\cos\psi} - e^{-jkh\cos\psi}$

$$= 2j\left(\frac{e^{jkh\cos\psi} - e^{-jkh\cos\psi}}{2j}\right)$$

$$AF = 2j\cdot\sin(kh\cos\psi)$$

$$\cos\psi = \sin\theta\cdot\cos\phi$$




$$AF = 2j\sin(kh\cos\phi\cdot\sin\theta) = 2j\sin\left(\frac{2\pi}{\lambda}\cdot\frac{\lambda}{20}\sin\theta\cdot\cos\phi\right) = 2j\sin\left(\frac{\pi}{10}\sin\theta\cdot\cos\phi\right)$$

b. $E_\theta = E_\theta(\text{single Wire})\cdot AF$

$$= 2j\sin\left(\frac{\pi}{10}\sin\theta\cdot\cos\phi\right)\left\{\sin\theta\frac{\sin\left[\frac{kl}{2}(\cos\theta - K)\right]}{\frac{kl}{2}(\cos\theta - K)}\right\} j\eta\frac{klI_0e^{-jkr}}{4\pi r}\cdot e^{-j\frac{kl}{2}(K-\cos\theta)}$$

c. $R_L = 138\log_{10}\left(4\frac{h}{d}\right)$

$$= 138\log_{10}\left(4\frac{\lambda/20}{\lambda/100}\right) = 138\cdot\log_{10}(4\cdot 5) = 138\cdot\log_{10}(20)$$

$$R_L = 138(1.3010) = 179.542$$

10-10. From (10-20) $\Rightarrow D_o = 2.94\left(\frac{\ell}{\lambda}\right) + 1.15$

a. $\ell = 2\lambda$; $D_o = 2.94(2) + 1.15 = 6.43 = 8.082\ dB$

b. $\ell = 3\lambda$; $D_o = 2.94(3) + 1.15 = 9.97 = 9.987 dB$

In Problem 10-7

$D_o(\ell = 2\lambda) = 6.984 = 8.44 dB$

$D_o(\ell = 3\lambda) = 9.4628 = 9.76\ dB$

10-11. $D_o = 8dB = 10 \cdot \log_{10} D_o(dim.) \Rightarrow D_o(dim.) = 10^{0.8} = 6.3096$

From (10-20) $\Rightarrow D_o(dim.) = 2.94\left(\frac{\ell}{\lambda}\right) + 1.15 = 6.3096 \Rightarrow \frac{\ell}{\lambda} = 1.755$

$\ell = 1.755\ \lambda$

Using (10-19b) $\Rightarrow 2\theta_o = 13.39\left(\frac{\ell}{\lambda}\right)^2 - 78.27\left(\frac{\ell}{\lambda}\right) + 169.77$

$2\theta_o = 13.39(1.755)^2 - 78.27(1.755) + 169.77$

$2\theta_o \simeq 73.65°$

10-12. $D_o = 5dB = 10 \log_{10} D_o(dim.) \Rightarrow D_o(dim.) = 10^{0.5} = 3.1623$

From (10-20) $\Rightarrow D_o(dim.) = 2.94\left(\frac{\ell}{\lambda}\right) + 1.15 = 3.1623 \Rightarrow \frac{\ell}{\lambda} = 0.6844$, $\ell = 0.6844$

Using (10-19a) $\Rightarrow 2\theta_o = -149.3\left(\frac{\ell}{\lambda}\right)^3 + 603.4\left(\frac{\ell}{\lambda}\right)^2 - 809.5\left(\frac{\ell}{\lambda}\right) + 443.6$

$2\theta_o = -149.3(0.6844)^3 + 603.4(0.6844)^2 - 809.5(0.6844) + 443.6$

$2\theta_o \simeq 124.34°$

10-13. a. $2\theta_o = -149.3\left(\frac{\ell}{\lambda}\right)^3 + 603.4\left(\frac{\ell}{\lambda}\right)^2 - 809.5\left(\frac{\ell}{\lambda}\right) + 443.6$

$= -149.3(0.5)^3 + 603.4(0.5)^2 - 809.5(0.5) + 443.6$

$2\theta_o = 171.0375°$

b. $D_o = 2.94\left(\frac{\ell}{\lambda}\right) + 1.15$

$= 2.94(0.5) + 1.15$

$D_o = 1.47 + 1.15 = 2.62 = 4.183\ dB$

c. $G_{To} = \varepsilon_t D_o = 0.35(2.62) = 0.917$

$G_{To} = 0.917 = -0.376\ dB$

10-14. $D_o = 9\,dB = 10^{0.9} = 7.9433$

a. $7.9433 = 2.94\left(\frac{l}{\lambda}\right) + 1.15$

$$\frac{l}{\lambda} = \frac{7.9433 - 1.15}{2.94} = 2.3106 \Rightarrow l = 2.3106\lambda$$

b. $2\theta_o = 13.39\left(\frac{l}{\lambda}\right)^2 - 78.27\left(\frac{l}{\lambda}\right) + 169.77$

$= 13.39(2.3106)^2 - 78.27(2.3106) + 169.77$

$2\theta_o = 71.49 - 180.8507 + 169.77 = 60.41°$

$2\theta_o = 60.41°$

c. $D_t = D_o D_a$

$D_a = 2N\left(\frac{d}{\lambda}\right) = 2 \cdot 10 \cdot \left(\frac{\lambda}{4\lambda}\right) = 5 = 6.989\,dB \simeq 7\,dB$

$D_t = 7.9433(5) = 39.7165 = 15.9897\,dB \simeq 16\,dB$

$D_t(dB) = D_o(dB) + D_a(dB) = 9 + 7 = 16\,dB$

10-15. $h_1 + h_2 = 0.25\lambda$, $a = 0.25 \times 10^{-3}\lambda$

a. Using Fig. 10-11(b) $\Rightarrow h_1 \simeq 0.14\lambda$

Therefore the bent is made at $h_1 \simeq 0.14\lambda$ above the ground.

b. Using Fig. 10-11(b) $\Rightarrow R = 24$ ohms

c. $\Gamma = \left|\frac{R - Z_o}{R + Z_o}\right| = \left|\frac{24 - 50}{24 + 50}\right| = 0.35135$

$$VSWR = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.35135}{1 - 0.35135} = 2.083$$

10-16. $S = \lambda/50$, $\lambda = 3 \times 10^8 / 4 \times 10^8 = \frac{3}{4} = 0.75$ m

a. Using (10-28a) $\Rightarrow C = \pi D = \sqrt{2\lambda(\lambda/50)} = \lambda/5 = 0.2\lambda$

$C = 0.2\lambda = 0.15$ m, $D = 0.2\lambda/\pi = 0.06367\lambda = 0.04775$ m

b. $L_o = \sqrt{S^2 + C^2} = \sqrt{\left(\frac{\lambda}{50}\right)^2 + \left(\frac{\lambda}{5}\right)^2} = \lambda\sqrt{\frac{1}{2500} + \frac{1}{25}} = 0.2009975\lambda = 0.150748$ m

c. $L_n = NL_o = 5(0.2009975\lambda) = 1.0049875\lambda = 0.75374$ m

d. $\alpha = \tan^{-1}\left(\frac{S}{\pi D}\right) = \tan^{-1}\left(\frac{S}{C}\right) = \tan^{-1}\left(\frac{\frac{\lambda}{50}}{\frac{\lambda}{5}}\right) = \tan^{-1}\left(\frac{1}{10}\right) = 5.71°$

10-17. $f = 300\,\text{MHz} \Rightarrow \lambda = 300\times10^6/300\times10^6 = 1\,\text{m}$

a. Near optimum $\Rightarrow C \simeq \lambda = 1\,\text{m}$

b. Pitch angle near optimum is $\alpha \simeq 14^\circ$

Using (10-24) $\Rightarrow \alpha = \tan^{-1}\left(\frac{S}{C}\right) \Rightarrow \frac{S}{C} = \tan\alpha = \tan(14^\circ) = 0.24933$

$S = 0.24933\,C = 0.24933\,\lambda = 0.24933\,\text{m}$

c. Using (10-30) $\Rightarrow R_{in} \simeq 140\left(\frac{C}{\lambda}\right) = 140$ ohms

d. Using (10-31) $\Rightarrow \text{HPBW(degrees)} \simeq \frac{52\lambda^{3/2}}{C\sqrt{NS}} = \frac{52(1)}{1\sqrt{5(0.24933)}} = 46.57^\circ$

Using (10-32) $\Rightarrow \text{FNBW(degrees)} \simeq \frac{115\lambda^{3/2}}{C\sqrt{NS}} = \frac{115(1)}{1\sqrt{5\cdot(0.24933)}} = 102.997^\circ$

Using (10-33) $\Rightarrow D_o \simeq 15N\frac{C^2S}{\lambda^3} = 15\cdot(5)\cdot\frac{1(0.24933)}{1}$

$= 18.69975 = 12.718\,\text{dB}$

AR (for increased directivity) $= (2N+1)/(2N) = \frac{2(5)+1}{2(5)} = 1.1$

e. $Z_o = 50$;

$|\Gamma| = \left|\frac{R_{in}-Z_o}{R_{in}+Z_o}\right| = \left|\frac{140-50}{140+50}\right| = 0.47368$

$\text{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.47368}{1-0.47368} \simeq 2.8$

$Z_o = 75$;

$|\Gamma| = \frac{R_{in}-Z_o}{R_{in}+Z_o} = \frac{140-75}{140+75} = 0.302326$

$\text{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.302326}{1-0.302326} \simeq 1,867$

10-18. The normalized far-field pattern is

$$E = \sin\left(\frac{\pi}{2N}\right)\cdot\cos\theta\cdot\frac{\sin[(N/2)\psi]}{\sin[\psi/2]},$$

where $\psi = k_o(S\cos\theta - \frac{L}{P})$, $\quad (\leftarrow L = NS = 5\cdot(0.24933) = 1.24665)$

$L = 1.24665\,\text{m}$

a. $P = \frac{L/\lambda_o}{S/\lambda_o + 1} = \frac{1.24665}{0.24933 + 1} = 0.99785.$ $\quad \lambda = 1\,\text{m},$

b. $P = \frac{L/\lambda_o}{S/\lambda_o + \left(\frac{2N+1}{2N}\right)} = \frac{1.24665}{0.24933 + 1.1} = 0.9239$

c. $P = 1.$

(Continued)

10-18 (Cont'd)

Helical Antenna ( Ordinary End-Fire )

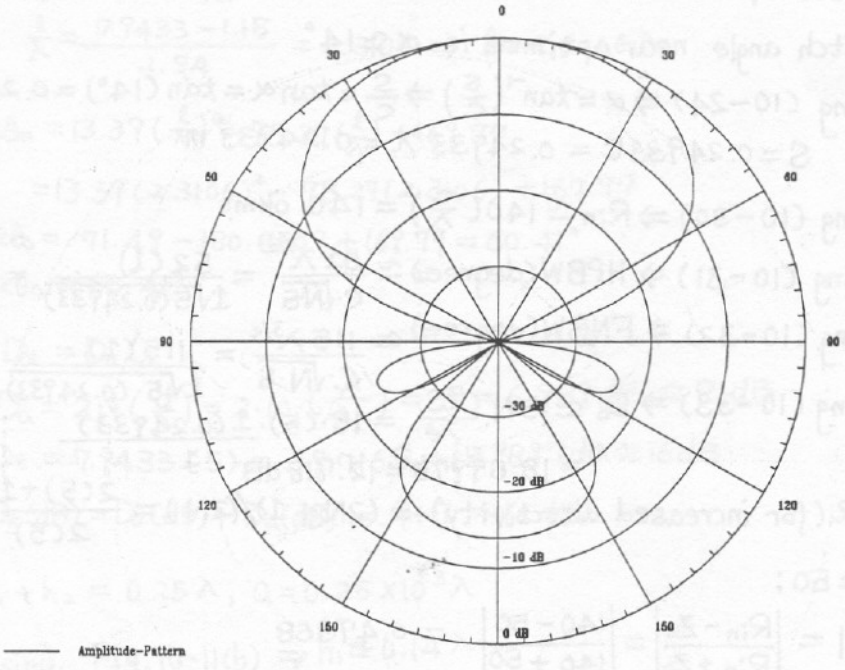



Fig. P 10.18 (a)

Helical Antenna ( Hansen_Woodyard End-Fire )

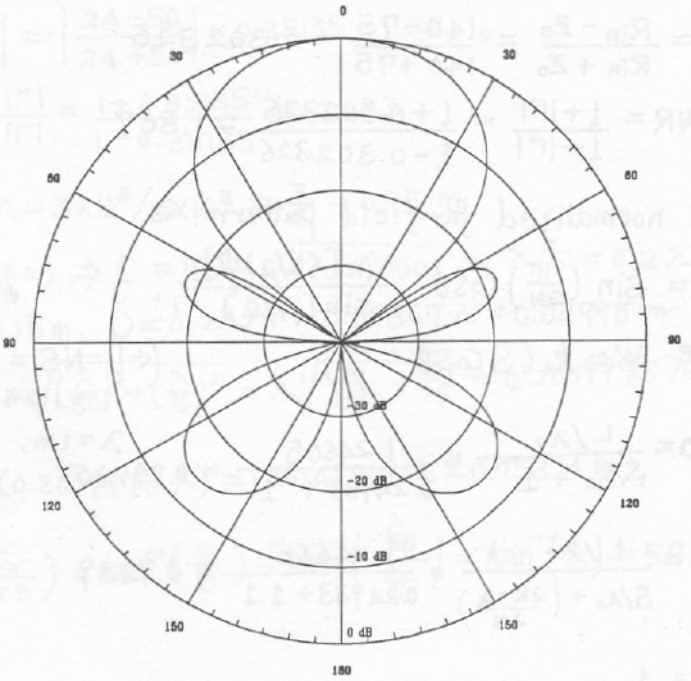



Fig. P 10.18 (b)

(continued)

10-18 (Cont'd)

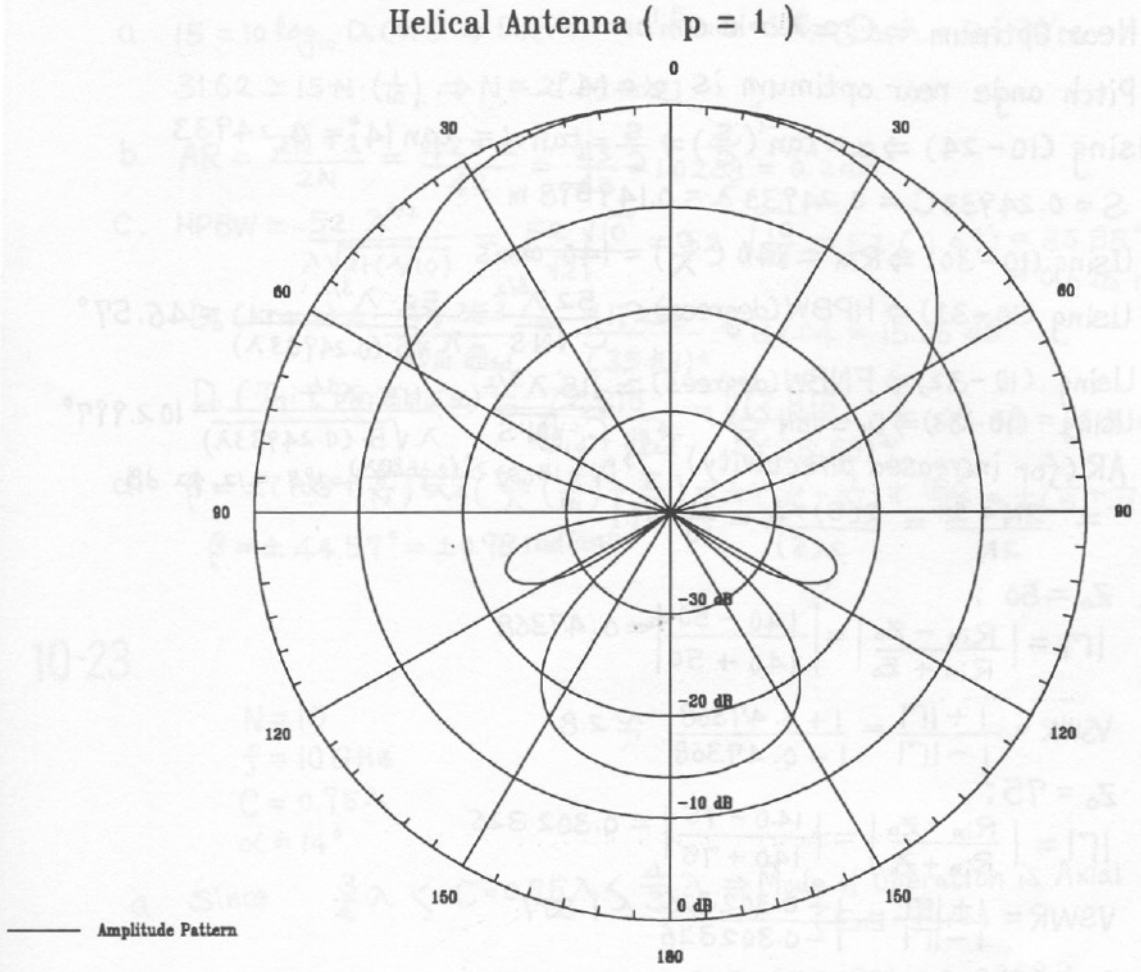



Fig. P10.8 (c)

10-19. Using computer Program Directivity at the end of Chapter 2.

a. Ordinary end-fire; $D_o = 9.2301$ dB

b. Hansen-Woodyard end-fire; $D_o = 11.99413$ dB

c. $p=1$ ; $D_o = 9.14576$ dB

10-20. $f = 500\ MHz \Rightarrow \lambda = 300\times10^6/500\times10^6 = 3/5 = 0.6\ m$

a. Near Optimum $\Rightarrow C \simeq \lambda = 0.6\ m$

b. Pitch angle near optimum is $\alpha = 14°$.

Using (10-24) $\Rightarrow \alpha = \tan^{-1}\left(\frac{S}{C}\right) \Rightarrow \frac{S}{C} = \tan\alpha = \tan 14° = 0.24933$

$S = 0.24933\,C = 0.24933\,\lambda = 0.149598\ m$

c. Using (10-30) $\Rightarrow R_{in} \simeq 140\left(\frac{C}{\lambda}\right) = 140$ ohms

Using (10-31) $\Rightarrow$ HPBW (degrees) $\simeq \frac{52\lambda^{3/2}}{C\cdot\sqrt{N\cdot S}} = \frac{52\cdot\lambda^{3/2}}{\lambda\sqrt{5\cdot(0.24933\lambda)}} = 46.57°$

Using (10-32) $\Rightarrow$ FNBW (degrees) $\simeq \frac{115\lambda^{3/2}}{C\cdot\sqrt{N\cdot S}} = \frac{115\lambda^{3/2}}{\lambda\sqrt{5\cdot(0.24933\lambda)}} = 102.997°$

Using (10-33) $\Rightarrow D_o \simeq 15N\frac{C^2S}{\lambda^3}$ $\rightarrow D_o = 15(5)\frac{\lambda^2(0.24933\lambda)}{\lambda^3} = 18.7 = 12.72\ dB$

AR (for increased directivity) $= \frac{2N+1}{2N} = \frac{2(5)+1}{2(5)} = \frac{11}{10} = 1.1$

e, $Z_o = 50$ ;

$|\Gamma| = \left|\frac{R_{in}-Z_o}{R_{in}+Z_o}\right| = \left|\frac{140-50}{140+50}\right| = 0.47368$

$VSWR = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.47368}{1-0.47368} \simeq 2.8$

$Z_o = 75$;

$|\Gamma| = \left|\frac{R_{in}-Z_o}{R_{in}+Z_o}\right| = \left|\frac{140-75}{140+75}\right| = 0.302326$

$VSWR = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.302326}{1-0.302326} \simeq 1.867$

10-21. $f = 500\ MHz \rightarrow \lambda = 60\ cm$

a. $\alpha = \tan^{-1}\left(\frac{S}{C}\right) \Rightarrow S = C\cdot\tan\alpha = 60\cdot\tan(13°) = 60\cdot(0.23) = 13.852\ cm$

$HPBW\ (degrees) = 45° = \frac{52\cdot\lambda^{3/2}}{C\sqrt{NS}} = \frac{52\cdot(60)^{3/2}}{60\sqrt{N\cdot(13.852)}} = \frac{52(464.758)}{60\cdot\sqrt{N\cdot(13.852)}}$

$N = \frac{1}{13.852}\left[\frac{52\cdot(464.758)}{60(45)}\right]^2 = \frac{80.1185}{13.852} = 5.7839 \simeq 6$

b. $D_o = 15\cdot N\cdot\frac{C^2S}{\lambda^3} = 15\cdot(6)\cdot\frac{(60)^2\cdot(13.852)}{(60)^3} = \frac{90\cdot(13.852)}{60} = 20.778 = 13.176\,dB$

c. $AR = \frac{2N+1}{2N} = \frac{2(6)+1}{12} = \frac{13}{12} = 1.0833$

d. $C_1 = \frac{3}{4}\lambda = 60\ cm \Rightarrow \lambda_1 = \frac{60(4)}{3} = 80\ cm \Rightarrow \lambda_1 = \frac{v}{f_1}$

$f_1 = 3.\times10^{10}/80 = 375\ MHz$

$C_2 = \frac{4}{3}\lambda_2 = 60\ cm \Rightarrow \lambda_2 = \frac{3\cdot(60)}{4} = 45\ cm$, $f_2 = 666.67\ MHz$

e. $R_o = 140(1) = 140$ ohms, $R_1 = 140\left(\frac{3}{4}\right) = 105$ ohms

$R_2 = 140\left(\frac{4}{3}\right) = 186.67$ ohms

10-22. $D_o \simeq 15N \cdot \frac{C^2 S}{\lambda^3}$

a. $15 = 10\log_{10} D_o(VR) \Rightarrow D_o(VR) = 10^{1.5} = 31.62$, $C \simeq \lambda$, $S = \lambda/10$

$31.62 \simeq 15 \cdot N \cdot (\frac{1}{10}) \Rightarrow N = 21.08 \simeq 21$

b. $AR = \frac{2N+1}{2N} = \frac{42+1}{42} = \frac{43}{42} = 1.0238 = 0.2\,dB$

c. $HPBW = \frac{52 \cdot \lambda^{3/2}}{\lambda\sqrt{21 \cdot (\lambda/10)}} = \frac{52 \cdot \sqrt{10}}{\sqrt{21}} = 52\sqrt{\frac{10}{21}} = 52 \cdot (0.69) = 35.88^\circ$
$= 0.626$ rad

$D_o(\text{kraus}) = \frac{41,253}{\Theta_{1d}\,\Theta_{2d}} = \frac{41,253}{(35.88)^2} = 32.04 = 15.06\,dB$

$D_o(\text{Tai \& Pereira's}) = \frac{72,815}{\Theta_{1d}^2 + \Theta_{2d}^2} = \frac{72,815}{2(35.88)^2} = 28.28 = 14.51\,dB$

d. $\beta = \pm(kS + \frac{\pi}{N}) = \pm(\frac{2\pi}{\lambda}(\frac{\lambda}{10}) + \frac{\pi}{21}) = \pm(\frac{1}{5} + \frac{1}{21})\pi \cdot \frac{180^\circ}{\pi} = \pm(\frac{1}{5} + \frac{1}{21}) \cdot 180^\circ$

$\beta = \pm 44.57^\circ = \pm 0.78$ radians

10-23.

$N = 10$
$f = 10\,GHz$
$C = 0.95\lambda$
$\alpha = 14^\circ$

a. Since $\frac{3}{4}\cdot\lambda < C = 0.95\lambda < \frac{4}{3}\lambda \Rightarrow$ Mode of Operation is Axial (End-Fire)

b. $S = C \cdot \tan\alpha = 0.95 \cdot \lambda \cdot \tan(14^\circ) = 0.95 \cdot \lambda \cdot (0.2493) = 0.2369\lambda$

$HPBW = \frac{52 \cdot \lambda^{3/2}}{0.95 \cdot \lambda \cdot \sqrt{10 \cdot (0.2369\lambda)}} = 35.57^\circ = \frac{52 \cdot \lambda^{3/2}}{C\sqrt{N \cdot S}}$

c. $D_o = 15N \cdot \frac{C^2 S}{\lambda^3} = 15 \cdot (10) \cdot \frac{(0.95\lambda)^2 (0.2369\lambda)}{\lambda^3} = 32.07 = 15.06\,dB$

$D_o \simeq \frac{41,253}{\Theta_{1d}\,\Theta_{2d}} = \frac{41,253}{(35.57)^2} = 32.61 = 15.13\,dB$

10-24 $N = 10$ $f = 500\,MHz$

$C = \lambda$ $P_{in} = 5$ Watts

$Z_o = 78$

a. $R \simeq 140\left(\frac{C}{\lambda}\right) = 140$ ohms

$\Gamma = \frac{R - Z_o}{R + Z_o} = \frac{140 - 78}{140 + 78} = 0.2844$

$P_{rad} = (1 - |\Gamma|^2) \cdot P_{in} = (1 - |0.2844|^2) \cdot 5 = 0.919116 \cdot (5) = 4.595$

$P_{rad} = 4.595$ watts

b. $W_{isotropic} \simeq \frac{P_{rad}}{4\pi r^2} = \frac{4.595}{4\pi (10^4)^2} = 0.3656 \times 10^{-8}$

$W_{isotropic} = 3.656 \times 10^{-9} = 3.656$ n watts/m²

c. $D_o \simeq 15 \cdot N \cdot \frac{C^2 \cdot S}{\lambda^3} = 15 \cdot (10) \cdot \frac{\lambda^2 \cdot \lambda}{10 \cdot \lambda^3} = 15 = 11.76\,dB$

$W_{helix}\Big|_{max} = W_{isotropic} \cdot D_o = 3.656 \times 15 \times 10^{-9} = 54.84 \times 10^{-9}$

$W_{helix}\Big|_{max} = 54.84$ n watts/m²

d. $A_{em} = \frac{\lambda^2}{4\pi} \cdot D_o$ , $\lambda = \frac{v}{f} = \frac{3 \times 10^8}{5 \times 10^8} = 0.6$ meters

$A_{em} = \frac{0.36}{4\pi}(15) = 0.4297\ m^2$.

$P_{rec} = W_{helix} \cdot A_{em} = 54.84 \times 10^{-9}\,(0.4297) = 23.56 \times 10^{-9}$

$P_{rec} = 23.56 \times 10^{-9}$ Watts $= 23.56$ n watts

10-25

$$AR = \frac{2N+1}{2N} \simeq \frac{40+1}{40} = 1.025 \simeq 1$$

10-26

$$PLF = \left| \hat{a}_x \cdot \left( \frac{\hat{a}_x \pm j\,\hat{a}_y}{\sqrt{2}} \right) \right|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2} = -3.01\,dB$$

$VSWR = \frac{1+|\Gamma|}{1-|\Gamma|} = 1.1$ , $1 + |\Gamma| = 1.1(1 - |\Gamma|)$, $|\Gamma| = \frac{1.1 - 1}{1.1 + 1} = \frac{0.1}{2.1} = 0.0476$

a. $|\Gamma| = \left| \frac{\pm(Z_L - Z_c)}{Z_L + Z_c} \right| = \frac{Z_L - 300}{Z_L + 300}$ , $|\Gamma|(Z_L + 300) = \pm(Z_L - 300)$

$Z_L(|\Gamma| \mp 1) = 300(\mp 1 - |\Gamma|)$, $Z_L = 300\left(\frac{-|\Gamma| \mp 1}{|\Gamma| \mp 1}\right) = 300\left(\frac{-0.0476 \mp 1}{0.0476 \mp 1}\right)$

$$Z_L = \begin{cases} 300\left(\frac{-0.0476 - 1}{0.0476 - 1}\right) = 300(1.0958) = 328.74 \\ + \\ 300\left(\frac{-0.0476 + 1}{0.0476 + 1}\right) = 300(0.909) = 272.74 \end{cases}$$

b. $\frac{328.74}{4} = 82.186$ , $\frac{272.74}{4} = 68.185$

$\lambda/2$ Resonant folded dipole will transform

10-27 $U(\theta,\phi) \sim |E_\theta|^2 = \sin^2\theta \left| 1 - e^{-j\frac{\pi}{8}} e^{-jkS_{12}\sin\theta\sin\phi} - e^{j\frac{\pi}{8}} e^{jkS_{23}\sin\theta\sin\phi} \right|^2$

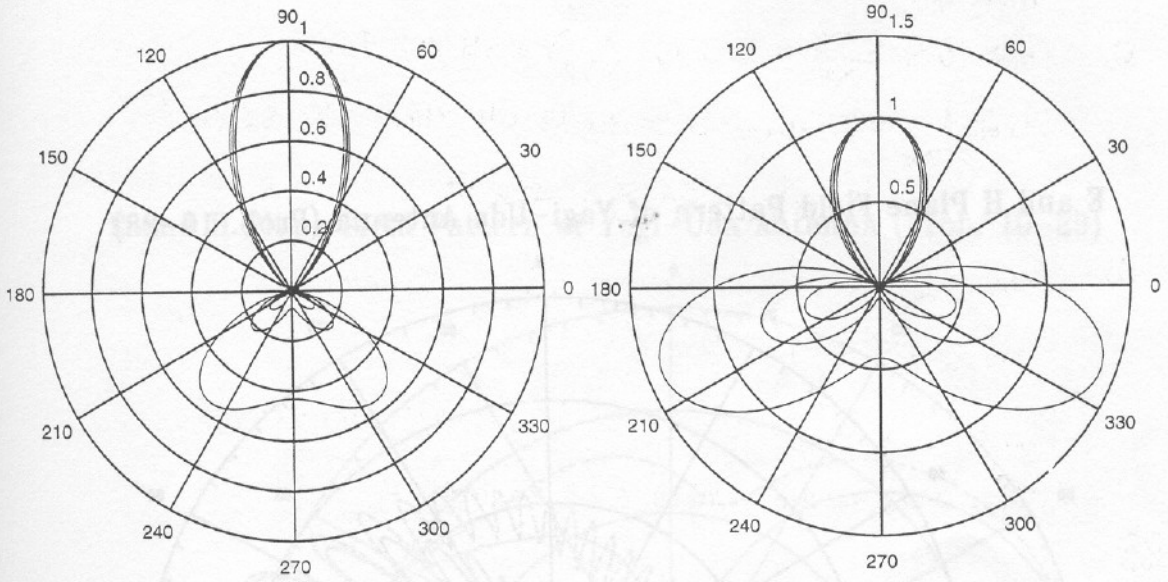
$U_{max} = \left| 1 - e^{-j\frac{\pi}{8}} e^{-jkS_{12}} - e^{j\frac{\pi}{8}} e^{jkS_{23}} \right|^2$

$$P_n(\theta,\phi) = \sin^2\theta \frac{\left| 1 - e^{-j\frac{\pi}{8}} e^{-jkS_{12}\sin\theta\sin\phi} - e^{j\frac{\pi}{8}} e^{jkS_{23}\sin\theta\sin\phi} \right|^2}{\left| 1 - e^{-j\frac{\pi}{8}} e^{-jkS_{12}} - e^{j\frac{\pi}{8}} e^{jkS_{23}} \right|^2}$$

E-plane : $\phi = \frac{\pi}{2}$ ; $0 \le \theta < 2\pi$

H-plane : $\theta = \frac{\pi}{2}$ ; $0 \le \theta < 2\pi$

a.




E-plane H-plane

b.

| $S_{23}$ | $FBR\|_{E\text{-plane}}$ |
|---|---|
| $0.15\lambda$ | 2.32 |
| $0.20\lambda$ | 15.0 |
| $0.25\lambda$ | 77.1 |

10-28 a. See Figure in below

b. 16.8385 dB

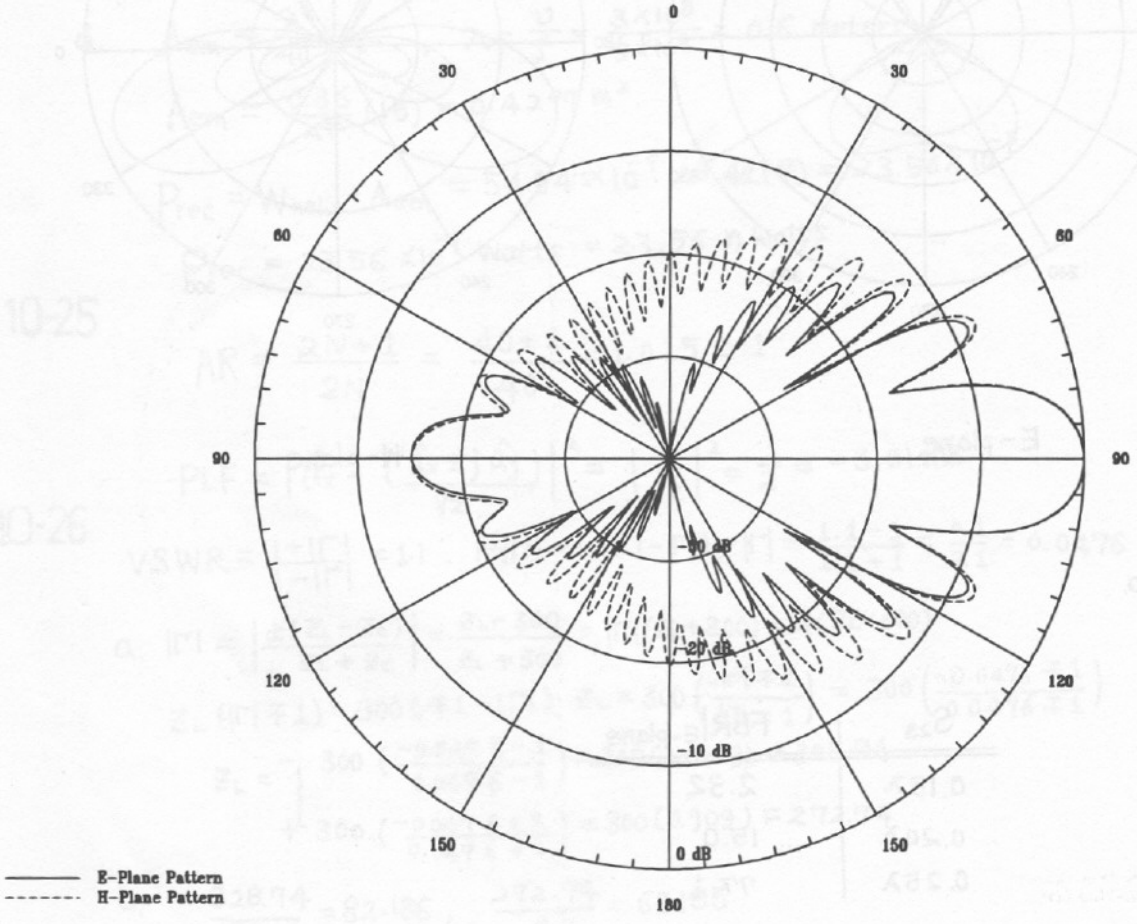
c. 17.6497 degrees

d. 17.9358 degrees

e. 15.2290 dB

f. 15.1964 dB

E and H Plane Field Pattern of Yagi-Uda Antenna (Prob. 10-28)



10-29 a. See Figure in below

b. 9.29577 dB

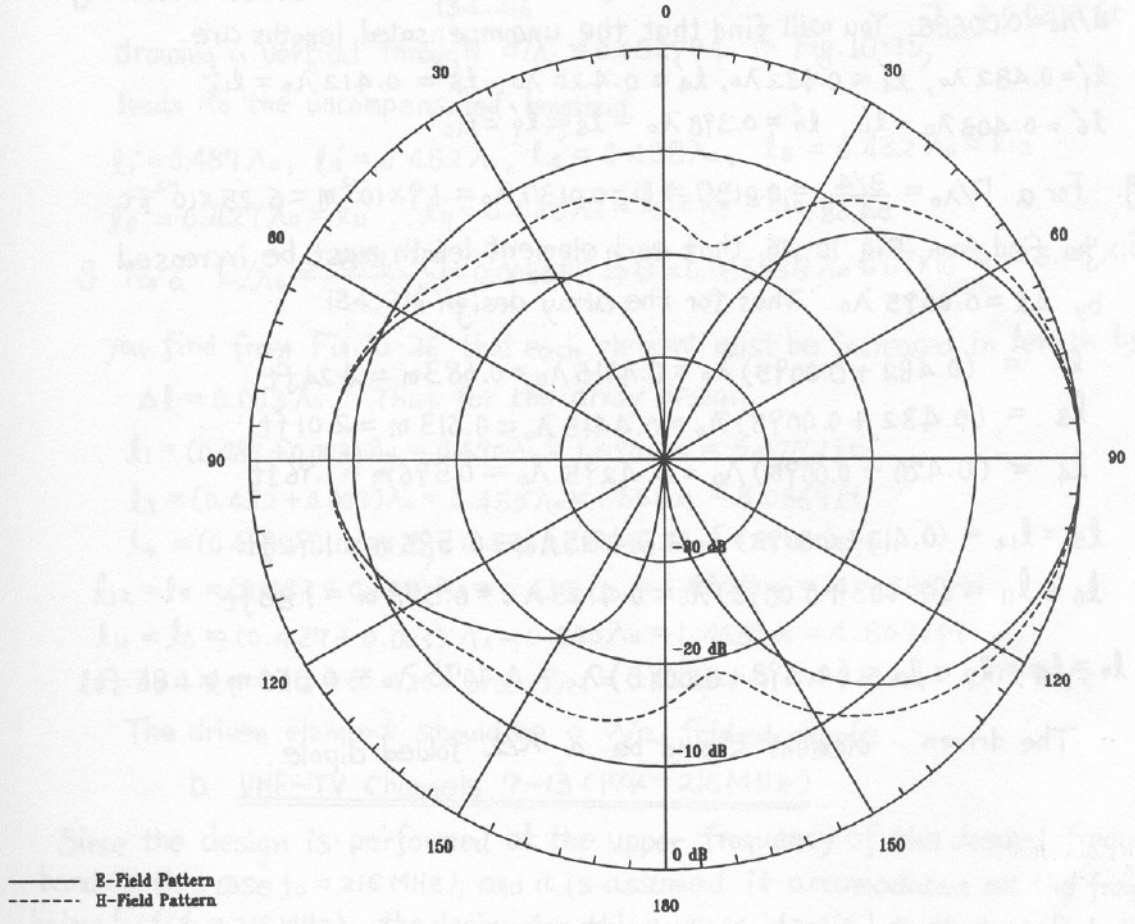
c. 54.0517 degrees

d. 65.7668 degrees

e. 7.65044 dB

f. 7.64547 dB

E and H Plane Field Pattern of Yagi–Uda Antenna (Prob. 10–29)



10-30. $\lambda_o = 3\times10^8/f_o = 3\times10^8/2.16\times10^8 = 1.3889\,m = 138.89\,cm = 54.68'' = 4.56'$

Since the gain of a $\lambda/2$ dipole is

$G_o = 10\log_{10}(1.643) = 2.156 \simeq 2.15\,dB$

then the gain of the Yagi-Uda array (above a $\lambda/2$ dipole) is

$G = 14.4 - 2.15 = 12.25\,dB$ (relative to a $\lambda/2$ dipole).

To accomplish this, according to Table 10-6, it would require

a. 12 elements (1 reflector, 1 feeder, 10 directors)

b. $S_{12}$ (reflector-feed) $= 0.2\lambda_o = 0.278\,m = 0.912\,ft$

c. $S_{ik}$ (director spacing) $= 0.2\lambda_o = 0.278\,m = 0.912\,ft$

d. $L = 2.2\lambda_o = 2.2(1.3889) = 3.0556\,m = 10.025\,ft$

e. For $d/\lambda_o = 0.0085$, $l_1'' = 0.482\lambda_o$, $l_3'' = 0.432\lambda_o$, $l_4'' = 0.415\lambda_o$

$l_5'' = 0.407\lambda_o = l_{12}''$, $l_6'' = 0.398\lambda_o = l_{11}''$, $l_7'' = 0.390\lambda_o = l_8'' = l_9'' = l_{10}''$

f. For a desired $d/\lambda_o = \frac{3/8}{54.68} = 0.00686 \Rightarrow d = 0.00686\lambda_o = 9.53\times10^{-3}\,m = 3.13\times10^{-2}\,ft$

Plot the lengths from part e in Fig. 10-25 and draw a vertical line through $d/\lambda_o = 0.00686$. You will find that the uncompensated lengths are

$l_1' = 0.482\lambda_o$, $l_3' = 0.432\lambda_o$, $l_4' = 0.420\lambda_o$, $l_5' = 0.412\lambda_o = l_{12}'$

$l_6' = 0.403\lambda_o = l_{11}'$, $l_7' = 0.398\lambda_o = l_8' = l_9' = l_{10}'$

g. For a $D/\lambda_o = \frac{3/4}{54.68} = 0.0137 \Rightarrow D = 0.0137\lambda_o = 1.9\times10^{-2}\,m = 6.25\times10^{-2}\,ft$

you find from Fig. 10-26. that each element length must be increased by $\Delta l = 0.0095\lambda_o$. Thus for the array design

$l_1 = (0.482 + 0.0095)\lambda_o = 0.4915\lambda_o = 0.683\,m = 2.24\,ft.$

$l_3 = (0.432 + 0.0095)\lambda_o = 0.4415\lambda_o = 0.613\,m = 2.01\,ft$

$l_4 = (0.420 + 0.0095)\lambda_o = 0.4295\lambda_o = 0.596\,m = 1.96\,ft$

$l_5 = l_{12} = (0.412 + 0.0095)\lambda_o = 0.4215\lambda_o = 0.585\,m = 1.92\,ft$

$l_6 = l_{11} = (0.403 + 0.0095)\lambda_o = 0.4125\lambda_o = 0.573\,m = 1.88\,ft$

$l_7 = l_8 = l_9 = l_{10} = (0.398 + 0.0095)\lambda_o = 0.4075\lambda_o = 0.566\,m = 1.86\,ft$

The driven element should be a $\lambda/2$ folded dipole.

10-31 a. VHF-TV Channels 2-6 (54-88 MHz)

$\lambda_0 = 3\times10^8/f_0 = 300\times10^6/88\times10^6 = 3.409\,m = 134.216'' = 11.18\,ft$

The gain & number of elements are the same as for Prob 10-30. Thus

Gain = G = 14.4 - 2.15 = 12.25dB (relative to a $\lambda/2$ dipole)

According to Table 10-6 and solution of Prob 10-30, the design requires

a. 12 elements (1 reflector, 1 feeder, 10 directors)

b. $S_{12}$ (reflector-feeder spacing) $= 0.2\lambda_0 = 0.6816\,m = 2.236\,ft$

c. $S_{ik}$ (director spacing) $= 0.2\lambda_0 = 0.6816\,m = 2.236\,ft$

d. $L = 2.2\lambda_0 = 2.2(3.409)\,m = 7.4998\,m = 24.6056\,ft$

e. For $d/\lambda_0 = 0.0085$

$\ell_1'' = 0.482\lambda_0$, $\ell_3'' = 0.432\lambda_0$, $\ell_4'' = 0.415\lambda_0$, $\ell_5'' = 0.407\lambda_0 = \ell_{12}''$

$\ell_6'' = 0.398\lambda_0 = \ell_{11}''$, $\ell_7'' = 0.390\lambda_0 = \ell_8'' = \ell_9'' = \ell_{10}''$

f. For a desired $d/\lambda_0 = \frac{3/8}{134.216} = 0.002794 \Rightarrow d = 0.002794\lambda_0 = 0.00952\,m = 0.0312\,ft$

drawing a vertical through $d/\lambda_0 = 0.002794$ in Fig. 10-25, leads to the uncompensated lengths of

$\ell_1' = 0.487\lambda_0$, $\ell_3' = 0.452\lambda_0$, $\ell_4' = 0.438\lambda_0$, $\ell_5' = 0.432\lambda_0 = \ell_{12}'$

$\ell_6' = 0.427\lambda_0 = \ell_{11}'$, $\ell_7' = 0.420\lambda_0 = \ell_8' = \ell_9' = \ell_{10}'$

g. For a $D/\lambda_0 = \frac{3/4}{134.216} = 0.005588 \Rightarrow D = 0.005588\lambda_0 = 1.9\times10^{-2}\,m = 6.25\times10^{-2}\,ft$

you find from Fig. 10-26 that each element must be increased in length by $\Delta\ell = 0.003\lambda_0$. Thus for the array design

$\ell_1 = (0.487 + 0.003)\lambda_0 = 0.490\lambda_0 = 1.6704\,m = 5.4782\,ft$

$\ell_3 = (0.452 + 0.003)\lambda_0 = 0.455\lambda_0 = 1.5511\,m = 5.0869\,ft$

$\ell_4 = (0.438 + 0.003)\lambda_0 = 0.441\lambda_0 = 1.5034\,m = 4.9304\,ft$

$\ell_{12} = \ell_5 = (0.432 + 0.003)\lambda_0 = 0.435\lambda_0 = 1.4829\,m = 4.8633\,ft$

$\ell_{11} = \ell_6 = (0.427 + 0.003)\lambda_0 = 0.430\lambda_0 = 1.4659\,m = 4.8074\,ft$

$\ell_7 = \ell_8 = \ell_9 = \ell_{10} = (0.420 + 0.003)\lambda_0 = 0.423\lambda_0 = 1.4420\,m = 4.7291\,ft$

The driven element should be a $\lambda/2$ folded dipole.

b. VHF-TV Channels 7-13 (174-216 MHz)

Since the design is performed at the upper frequency of the desired frequency band (in this case $f_0 = 216$ MHz), and it is assumed it accomodates all the frequencies below $f_0$ ($f < f_0 = 216$ MHz), the design for this part is identical to that of Prob. 10-30. For the specific details, see the solution of Problem 10-30.

10-32 $f_0 = 108\,\text{MHz} \Rightarrow \lambda_0 = \frac{300\times10^6}{108\times10^6} = 2.778\,\text{m} = 9.1134\,\text{ft}$

Gain = G = 12.35 − 2.15 = 10.2 dB (relative to a $\lambda/2$ dipole)

According to Table 10.6, the design requires

a. 6 elements (1 reflector, 1 feeder, 4 directors)

b. $S_{12}$ (reflector-feeder spacing) = 0.5556 m = 1.823 ft = $0.2\lambda_0$

c. $S_{ik}$ (director spacing) = $0.25\lambda_0 = 0.69445\,\text{m} = 2.278\,\text{ft}$

d. $L = 1.2\lambda_0 = 3.3336\,\text{m} = 10.936\,\text{ft}$

e. For $d/\lambda_0 = 0.0085$

$l_1'' = 0.482\lambda_0$, $l_3'' = 0.428\lambda_0 = l_6''$, $l_4'' = 0.420\lambda_0 = l_5''$

f. For a desired $d/\lambda_0 = \frac{3/8}{109.36} = 0.003429$

$\Rightarrow d = 0.003429\,\lambda_0 = 6.097\times10^{-3}\,\text{m} = 3.125\times10^{-2}\,\text{ft}$

drawing a vertical line through $d/\lambda_0 = 0.003429$ in Fig 10-25, leads to the uncompensated lengths of

$l_1' = 0.485\lambda_0$

$l_3' = 0.445\lambda_0 = l_6'$

$l_4' = 0.438\lambda_0 = l_5'$

g. For a $D/\lambda_0 = \frac{3/4}{109.36} = 0.00686 \Rightarrow D = 6.86\lambda_0 = 1.905\times10^{-2}\,\text{m} = 6.25\times10^{-2}\,\text{ft}$

You find from Fig. 10-26. that each element must be increased in length by $\Delta l = 0.004\lambda_0$. Thus for the array design

$l_1 = (0.485 + 0.004)\lambda_0 = 0.489\lambda_0 = 1.358\,\text{m} = 4.456\,\text{ft}$

$l_6 = l_3 = (0.445 + 0.004)\lambda_0 = 0.449\lambda_0 = 1.247\,\text{m} = 4.092\,\text{ft}$

$l_5 = l_4 = (0.438 + 0.004)\lambda_0 = 0.442\lambda_0 = 1.228\,\text{m} = 4.028\,\text{ft}$

The driven element should be a $\lambda/2$ folded dipole.

10-33. $f_o = 806\ \text{MHz} \Rightarrow \lambda_o = \frac{300\times10^6}{806\times10^6} = 0.3722\ \text{m} = 1.221\ \text{ft} = 14.654''$

Gain = G = 12.35 − 2.15 = 10.2 (relative to a $\lambda/2$ dipole).

According to Table 10.6, the design requires

a. 6 elements (1 reflector, 1 feeder, 4 directors)

b. $S_{12}$ (reflector-feed spacing) $= 0.20\lambda_o = 0.0744\text{m} = 0.2442\text{ft}$

c. $S_{ik}$ (director spacing) $= 0.25\lambda_o = 0.093\ \text{m} = 0.3053\ \text{ft}$

d. $L = 1.2\lambda_o = 0.44665\ \text{m} = 1.4654\ \text{ft}$

e. For $d/\lambda_o = 0.0085 \Rightarrow d = 8.5\times10^{-3}\lambda_o = 3.16\times10^{-3}\text{m} = 1.04\times10^{-2}\text{ft}$

$l_1'' = 0.482\lambda_o = 0.1794\ \text{m} = 0.5886\ \text{ft}$

$l_3'' = 0.428\lambda_o = 0.1593\ \text{m} = 0.5226\ \text{ft} = l_6''$

$l_4'' = 0.420\lambda_o = 0.1563\ \text{m} = 0.5129\ \text{ft} = l_5''$

f. For a desired $d/\lambda_o = \frac{3/32}{14.654} = 6.4\times10^{-3} \Rightarrow d = 0.00238\ \text{m} = 0.00782\ \text{ft}$, drawing a vertical line through $d/\lambda_o = 0.0064$ in Fig. 10-25, leads to the uncompensated lengths of

$l_1' = 0.484\ \lambda_o = 0.1801\ \text{m} = 0.591\ \text{ft}$

$l_3' = 0.434\ \lambda_o = 0.1615\ \text{m} = 0.530\ \text{ft} = l_6'$

$l_4' = 0.428\ \lambda_o = 0.1794\ \text{m} = 0.527\ \text{ft} = l_5'$

g. For a $D/\lambda_o = \frac{3/16}{14.654} = 0.012795 \Rightarrow D = 0.012795\lambda_o = 4.76\times10^{-3}\text{m} = 1.56\times10^{-2}\text{ft}$

You find from Fig 10-26 that each element must be increased in length by $\Delta l = 0.0076\lambda_o$. Thus for the array design

$l_1 = (0.484 + 0.0076)\lambda_o = 0.4916\lambda_o = 0.1830\ \text{m} = 0.6003\ \text{ft}$

$l_6 = l_3 = (0.434 + 0.0076)\lambda_o = 0.4416\lambda_o = 0.1644\ \text{m} = 0.5393\ \text{ft}$

$l_5 = l_4 = (0.428 + 0.0076)\lambda_o = 0.4356\lambda_o = 0.1621\ \text{m} = 0.5319\ \text{ft}$

The driven element should be a $\lambda/2$ folded dipole.

11-1. $\phi_0 = 0, \pi$ ; $f = 10$ MHz , $\rho_0 = 0.5 \times 10^{-3}\lambda$

a. $\rho_1 = \rho_0 e^{a(\phi-\phi_0)}$ , $L = (\rho_1 - \rho_0)\sqrt{1+\frac{1}{a^2}}$

$$\lambda = \frac{30\times10^9}{10\times10} = 3.000 \text{ cm} = 30 \text{ m} \Rightarrow L = \lambda = 30 \text{ m}$$

Thus

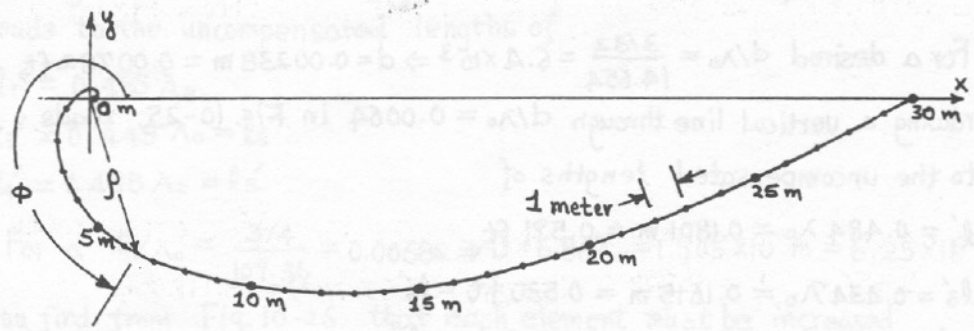
$$1\lambda = 0.5\times10^{-3}\lambda\,[e^{a(2\pi-0)} - 1]\sqrt{1+\frac{1}{a^2}}$$

Solving iteratively, it yields

$$a = 1.166 \text{ rad}^{-1}$$

b. $\rho_1 = \rho_0 e^{a(\phi-\phi_0)} = 0.5\times10^{-3}\lambda e^{1.166(2\pi-0)} = 0.75979\lambda = 22.79$ m

c. $\rho = \rho_0 e^{a(\phi-\phi_0)} = 0.015 e^{1.166(\phi-0)} = 0.015 e^{1.166\phi} = 0.5\lambda e^{1.166\phi}$




11-2. Using the geometry of Figure 11-9

$$\alpha = \tan^{-1}\left[\frac{\ell_n/2}{R_n}\right] = \tan^{-1}\left[\frac{\ell_n}{2R_n}\right]$$

Using (11-26a) $\Rightarrow \alpha = \tan^{-1}\left(\frac{\ell_n}{2R_n}\right) = \tan^{-1}\left(\frac{d_n/2\delta}{2R_n}\right) = \tan^{-1}\left(\frac{\frac{d_n}{R_n}}{4\delta}\right)$

$$\alpha = \tan^{-1}\left(\frac{\frac{d_n}{R_n}}{4\delta}\right) = \tan^{-1}\left(\frac{\frac{d_{n+1}}{R_{n+1}}}{4\delta}\right)$$

As the spacing between the elements approaches zero

$$\alpha = \tan^{-1}\left(\frac{\frac{d_{n+1}}{R_{n+1}}}{4\delta}\right) = \tan^{-1}\left[\frac{R_{n+1}-R_n}{\frac{R_{n+1}}{4\delta}}\right] = \tan^{-1}\left[\frac{1-\frac{R_n}{R_{n+1}}}{4\delta}\right] = \tan^{-1}\left(\frac{1-\tau}{4\delta}\right)$$

11-3. $D_o = 9dB$, $R_{in} = 75\Omega$

a. $f = (54-216)MHz$ $D_L = 1.905\,cm$, $D_s = 0.476\,cm$

Optimum $\sigma'|_{D_o=9dB} = 0.175$, $\tau|_{D_o=9dB} = 0.93$

$\alpha = \tan^{-1}\left(\frac{1-0.93}{4(0.175)}\right) = 5.711°$, $B_{ar} = 1.1 + 7.7(1-0.93)^2 \cot\alpha = 1.477$

$B_s = \frac{216}{54}(1.477) = 5.908$, $\lambda_{max} = \frac{3\times10^8}{54\times10^6} = 5.556\,m$

$L = \frac{\lambda_{max}}{4}\left(1-\frac{1}{B_s}\right)\cot\alpha = \frac{5.556}{4}\left(1-\frac{1}{5.908}\right)\cot(5.711°) = 11.538\,m$

$N = 1 + \frac{\ln(5.908)}{\ln(1/0.93)} = 25.477 \approx 25$ or 26 elements

$\delta' = \frac{0.175}{\sqrt{0.93}} = 0.1815$, $\ell_{max} = \frac{\lambda_{max}}{2} = 2.778\,m$, $d_{max} = 0.75in = 1.905\,cm$

$\ell_{max}/d_{max} = 145.8268$

Using (11-33) $Z_a = 120[\ln(145.8268) - 2.25] = 327.89$ ohms

$Z_a/R_{in} = 327.89/75 = 4.372 \Rightarrow Z_o/R_{in} \simeq 1.15$ (From Fig 11.14)

$Z_o = 1.15(75) = 86.25$ ohms

$S = \frac{3}{4}\cosh\left(\frac{86.25}{120}\right) = 0.75\cosh(0.719) = 0.75(1.26961) = 0.9522'' = 2.42\,cm$

b. Again from a.

$\tau = 0.93$, $\delta = 0.175$, $\alpha = 5.711°$, $B_{ar} = 1.477$, $B_s = \frac{88}{54}(1.477) = 2.407$

$\lambda_{max} = 5.556\,m$, $L = \frac{5.556}{4}\left(1-\frac{1}{2.407}\right)\cot(5.711°) = 8.1187\,m$

$N = 1 + \frac{\ln 2.407}{\ln(1/0.93)} = 13.10 \rightarrow N = 13$ elements

$\delta' = 0.1815$, $\ell_{max} = 2.778\,m$, $\frac{\ell_{max}}{d_{max}} = \frac{2.778}{1.905\times10^{-2}} = 145.8268$

$Z_a = 327.89$ ohm, $\frac{Z_a}{R_{in}} = 4.372 \Rightarrow \frac{Z_o}{R_{in}} = 1.15$, $Z_o = 1.15\cdot(75) = 86.25$ ohms

$S = \frac{3}{4}\cosh\left(\frac{86.25}{120}\right) = 0.75\cosh(0.719) = 0.75(1.26961) = 0.9522'' = 2.42\,cm$

c. Again from a

$\tau = 0.93$, $\delta = 0.175$, $\alpha = 5.711°$, $B_{ar} = 1.477$, $B_s = \frac{216}{174}(1.477) = 1.8335$

$\lambda_{max} = \frac{3\times10^8}{174\times10^6} = 1.724\,m$, $L = \frac{1.724}{4}\left(1-\frac{1}{1.8335}\right)\cot(5.711°) = 1.959\,m$

$N = 1 + \frac{\ln(1.8335)}{\ln(1/0.93)} = 9.354$, $N = 9$ or 10 element, $\frac{\ell_{max}}{d_{max}} = \frac{0.8612}{0.006} = 143.53$

$Z_a = 120[\ln(143.53) - 2.25] = 325.988$ ohms, $\frac{Z_a}{R_{in}} = 4.3465 \Rightarrow \frac{Z_o}{R_{in}} = 1.2$

$Z_o = 1.2(75) = 90$ ohms

$S = 0.6\cosh(90/120) = 0.6(1.2947) = 0.7768\,cm$ (Continued)

11-3 (Cont'd)

d. $f = (512-806)$ MHz, $D_L = 0.2$ cm, $D_s = 0.128$ cm

Again from a. $\sigma|_{D_o=9dB} = 0.175$, $\tau = 0.93$, $\alpha = 5.711°$, $Bar = 1.477$

$B_s = \frac{806}{512}(1.477) = 2.326$, $L = \left(\frac{3\times10^8/512\times10^6}{4}\right)\left(1-\frac{1}{2.326}\right)\cot\alpha = 0.835$ m

$N = 1 + \ln(2.326)/\ln(1/0.93) = 12.63 \approx N = 13$ elements

$l_{max} = \frac{\lambda_{max}}{2} = 0.293$ m, $\frac{l_{max}}{d_{max}} = \frac{0.293}{0.2\times10^{-2}} = 146.5$, $\frac{l_{min}}{d_{min}} = 145.3$, $\frac{l_n}{d_n} = 145.9$

$Z_a = 120[\ln(145.9) - 2.25] = 328.0$, $\frac{Z_a}{R_{in}} = \frac{328}{75} = 4.373$

$\sigma' = 0.1815$, From Fig 11.14 $\frac{Z_o}{R_{in}} = 1.15$ $\therefore Z_o = 1.15(75) = 86.25\ \Omega$

$S = (0.2\text{ cm})\cosh\left(\frac{86.25}{120}\right) = 2.539$ mm, $S = 2.54\times10^{-3}$ m

e. Again from a.

$B_s = \frac{108}{88}(1.477) = 1.8127$, $\lambda_{max} = \frac{3\times10^8}{88\times10^6} = 3.409$ m

$L = \frac{\lambda_{max}}{4}\left(1-\frac{1}{B_s}\right)\cot\alpha = 3.82073$ m, $N = 1 + \frac{\ln(1.8127)}{\ln(1/0.93)} = 9.1964$

$N = 9$ or 10 elements, $l_{max} = \frac{\lambda_{max}}{2} = 1.7045$ m, $\frac{l_{max}}{d_{max}} = \frac{1.7045}{1.169\times10^{-2}} = 145.8$

$Z_a = 120[\ln(145.808) - 2.25] = 327.875$ ohms

$\frac{Z_a}{R_{in}} = \frac{327.875}{75} = 4.37167 \Rightarrow \frac{Z_o}{R_{in}} \approx 1.16 \Rightarrow Z_o = 87$ ohm $\Omega$

$S = (1.169\text{ cm})\cosh\left(\frac{87}{120}\right) = 1.169(1.2745) = 1.48992$ cm

11-4. Since for all parts of Problem 11.3, $\tau = 0.93$, then according to (11-27) a. $\Delta = \ln(f_2) - \ln(f_1) = \ln\left(\frac{f_2}{f_1}\right) = \ln\left(\frac{1}{\tau}\right) = \ln\left(\frac{1}{0.93}\right) = 0.07257$

or $f_2/f_1 = 1/\tau = 1.07527 \Rightarrow f_2 = 1.07527 \cdot f_1$

b. $f_{max} = (1.07527)^N f_{min} \Rightarrow N = \log_{10}(f_{max}/f_{min})/\log_{10}(1.07527)$, Thus

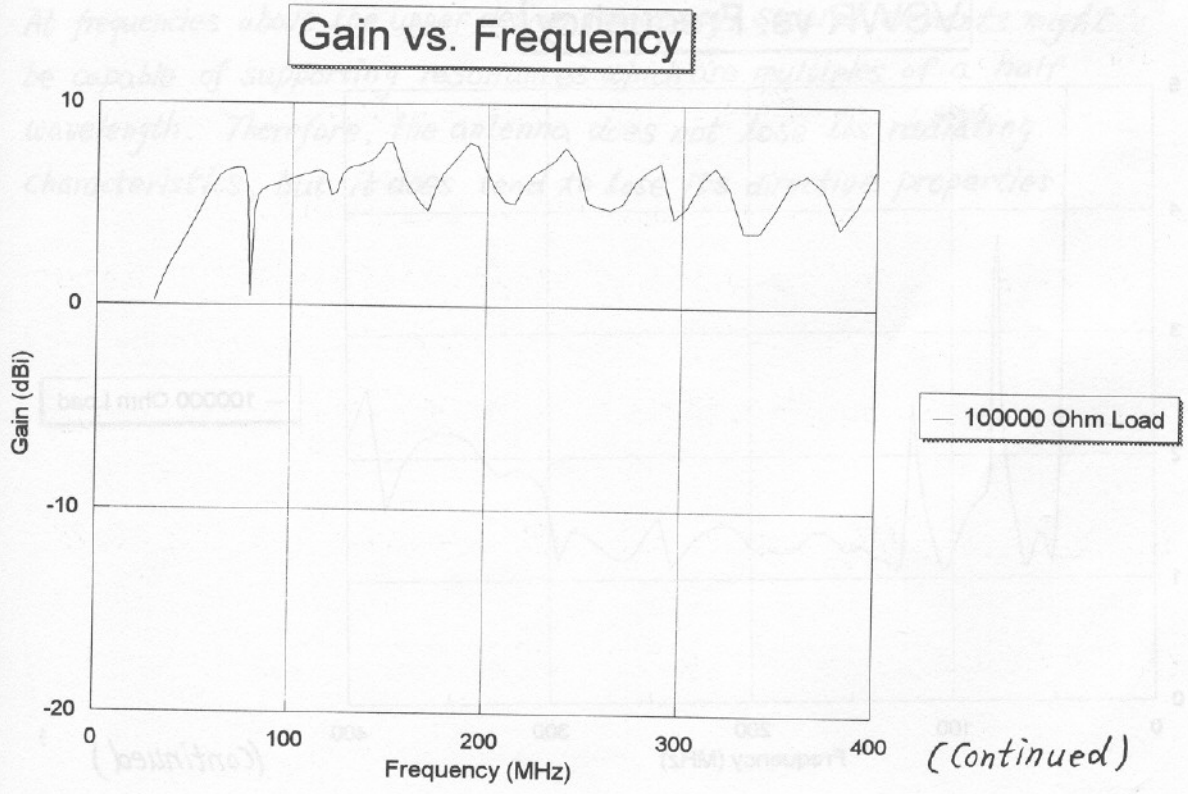
(a) $N = \log_{10}\left(\frac{216}{54}\right)/\log_{10}(1.07527) \approx 19$ (b) $N \simeq 6.729$, cc) $N \simeq 2.9795$

(d) $N \simeq 6.2526$ (e) $N \simeq 2.822$

11-5. Using the design program, LPOA.FOR, set up the main screen as shown. Whether or not the dipole diameters are quantized make little difference as long as they are quantized to reasonable value.

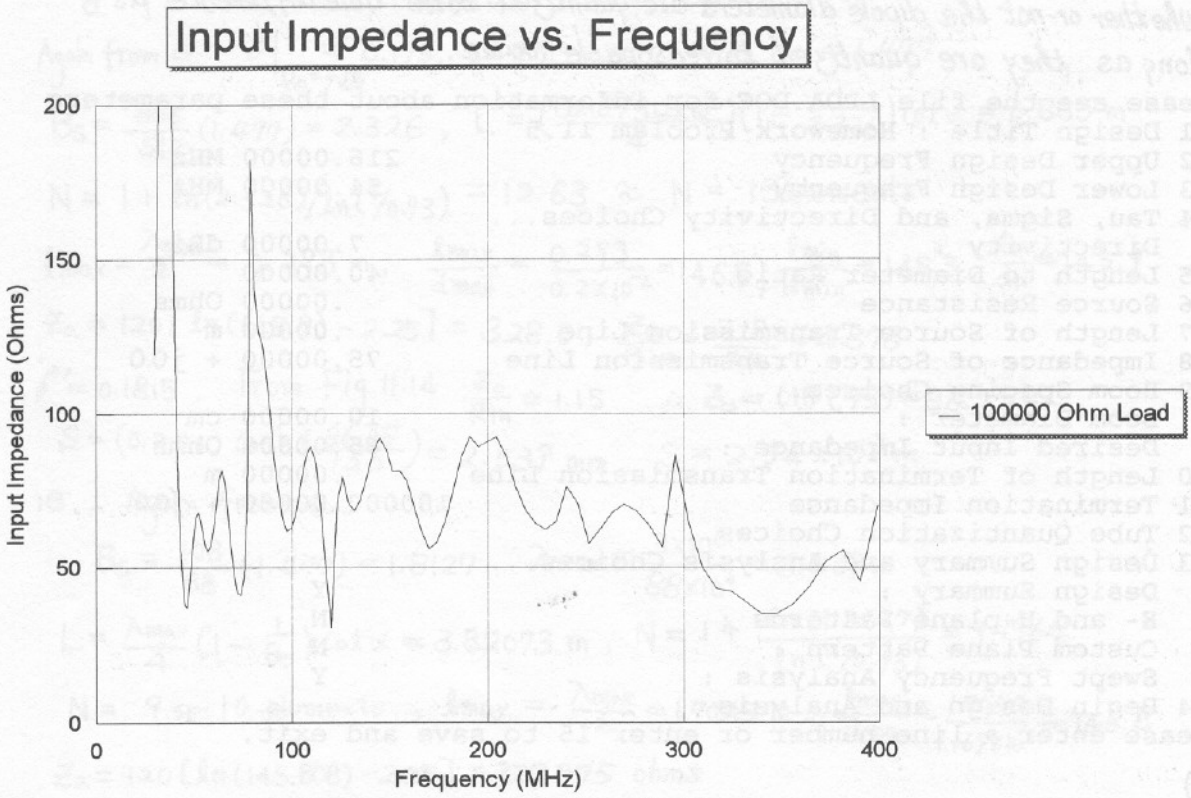
```
Please see the file LPDA.DOC for information about these parameters
  1 Design Title : Homework Problem 11.5
  2 Upper Design Frequency                    216.00000 MHz
  3 Lower Design Frequency                     54.00000 MHz
  4 Tau, Sigma, and Directivity Choices...
    Directivity :                               7.00000 dBi
  5 Length to Diameter Ratio                   40.00000
  6 Source Resistance                            .00000 Ohms
  7 Length of Source Transmission Line           .00000 m
  8 Impedance of Source Transmission Line      75.00000 + j0.0     .
  9 Boom Spacing Choices...
    Boom Diameter :                            10.00000 cm
    Desired Input Impedance :                  65.00000 Ohms
 10 Length of Termination Transmission Line        .00000 m
 11 Termination Impedance                  100000.00000 + j0.0     .(
 12 Tube Quantization Choices...
 13 Design Summary and Analysis Choices...
    Design Summary :                               Y
    E- and H-plane Patterns :                      N
    Custom Plane Pattern :                         N
    Swept Frequency Analysis :                     Y
 14 Begin Design and Analysis
Please enter a line number or enter 15 to save and exit.
```

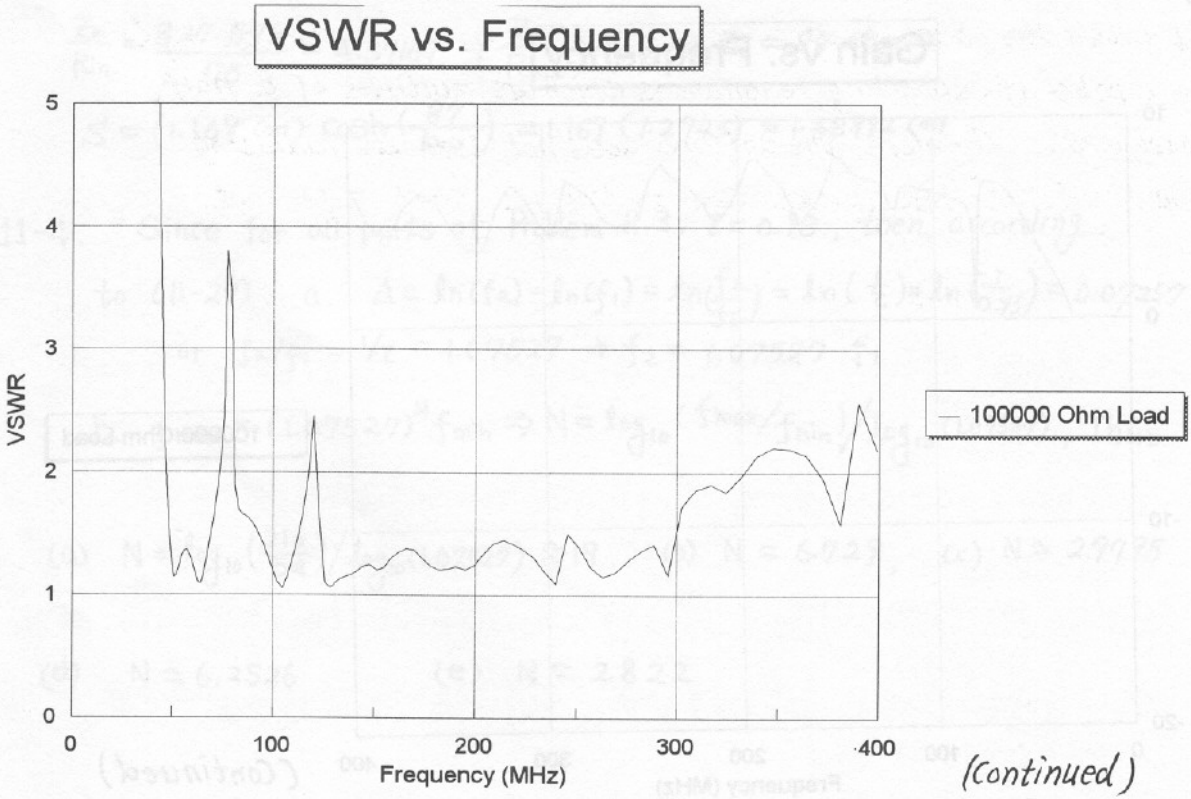
(a)




(Continued)

11-5 (Cont'd)




(continued)

11-5 (Cont'd)

(b) The directivity peaks when the frequency is shown below. The ratio of each frequency to the next highest is also shown

| f (MHz) | τ (Ratio) |
|---|---|
| 73.87 | 0.6447 |
| 114.58 | 0.7579 |
| 151.19 | 0.7937 |
| 190.49 | 0.7937 |
| 240.00 | 0.8313 |
| 288.72 | 0.8507 |
| 339.41 | 0.9330 |
| 363.77 | ——— |

The average of the ratios is 0.80. The computer program calculated 0.78

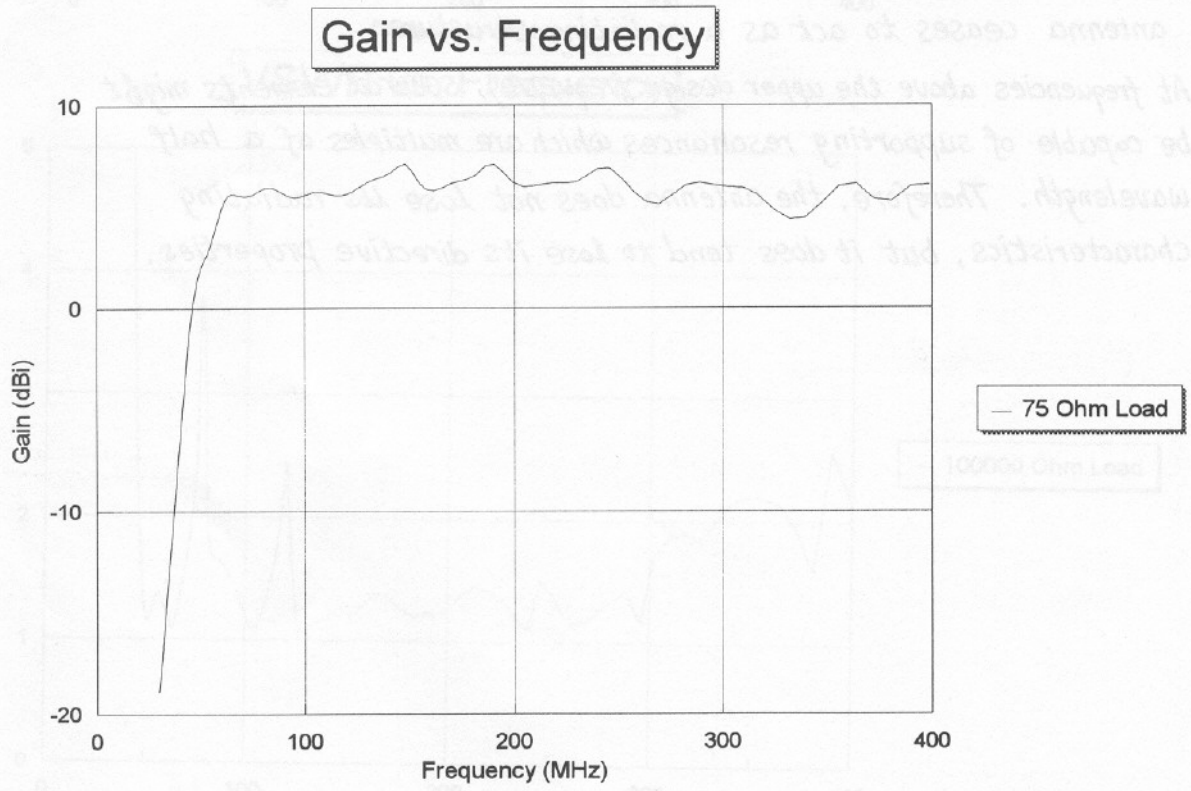
(c). At frequencies below the lower design frequency, no element is long enough to support a half-wavelength resonance, so the antenna ceases to act as a radiating structure.

At frequencies above the upper design frequency, several elements might be capable of supporting resonances which are multiples of a half wavelength. Therefore, the antenna does not lose its radiating characteristics, but it does tend to lose its directive properties.

11-6.

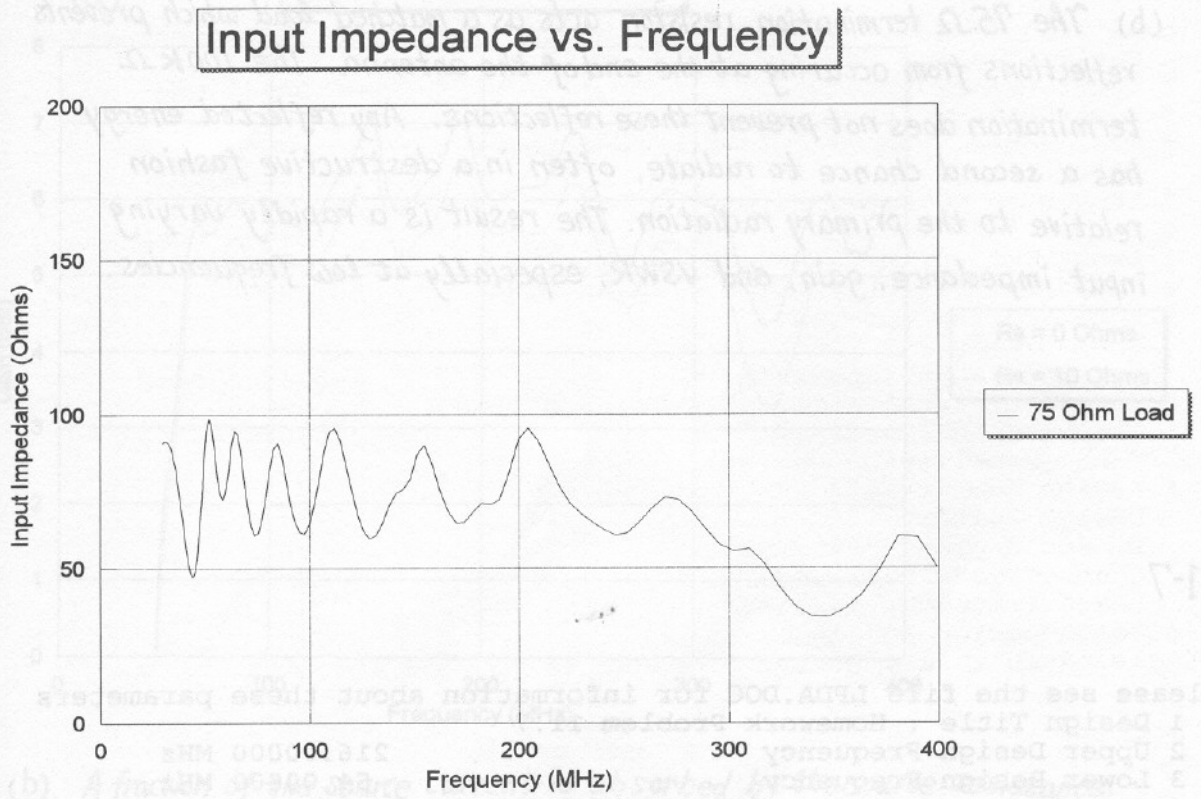
```
Please see the file LPDA.DOC for information about these parameters
  1 Design Title : Homework Problem 11.6
  2 Upper Design Frequency                    216.00000 MHz
  3 Lower Design Frequency                     54.00000 MHz
  4 Tau, Sigma, and Directivity Choices...
    Directivity :                               7.00000 dBi
  5 Length to Diameter Ratio                   40.00000
  6 Source Resistance                            .00000 Ohms
  7 Length of Source Transmission Line           .00000 m
  8 Impedance of Source Transmission Line      75.00000 + j0.0
  9 Boom Spacing Choices...
    Boom Diameter :                           10.00000 cm
    Desired Input Impedance :                 65.00000 Ohms
 10 Length of Termination Transmission Line       .00000 m
 11 Termination Impedance                     75.00000 + j0.0
 12 Tube Quantization Choices...
 13 Design Summary and Analysis Choices...
    Design Summary :                              Y
    E- and H-plane Patterns :                     N
    Custom Plane Pattern :                        N
    Swept Frequency Analysis :                    Y
 14 Begin Design and Analysis
Please enter a line number or enter 15 to save and exit.
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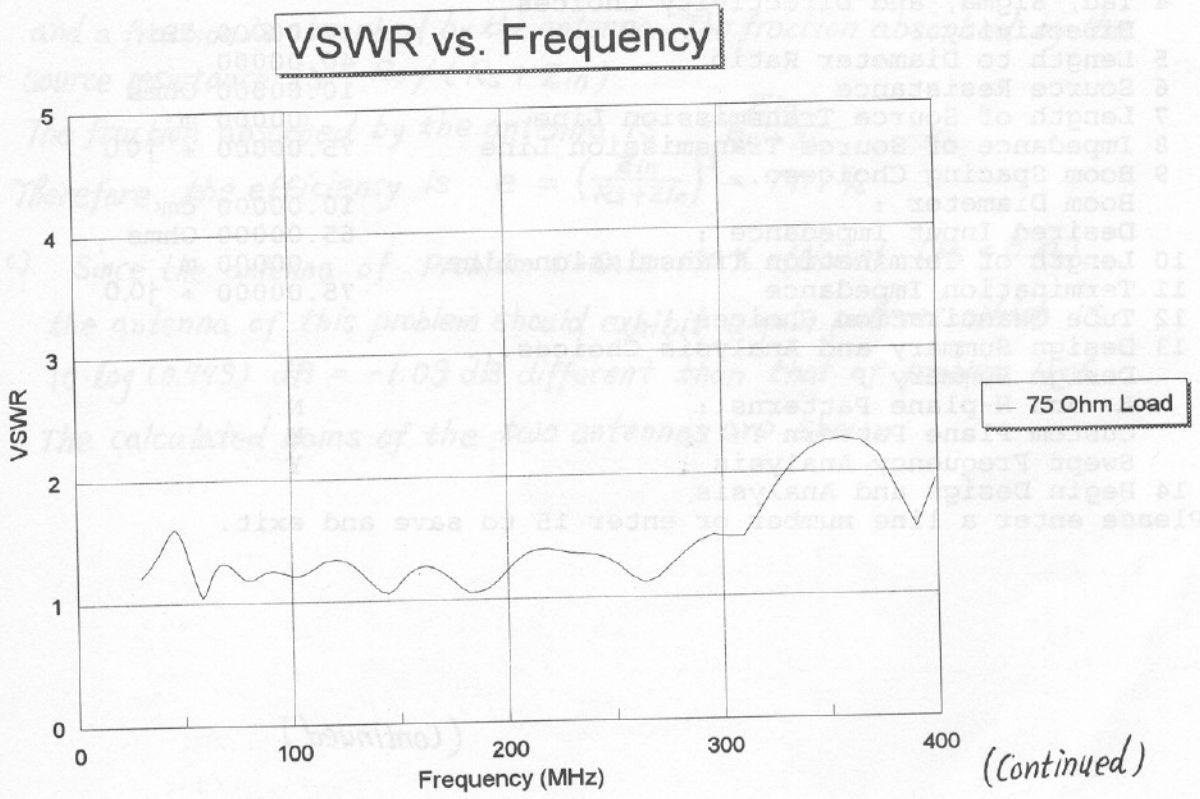
(a)




(Continued)

11-6. (Cont'd)




(Continued)

11-6 (Cont'd)

(b) The 75 Ω termination resistor acts as a matched load which prevents reflections from occuring at the end of the antenna. The 100 KΩ termination does not prevent these reflections. Any reflected energy has a second chance to radiate, often in a destructive fashion relative to the primary radiation. The result is a rapidly varying input impedance, gain, and VSWR, especially at low frequencies.

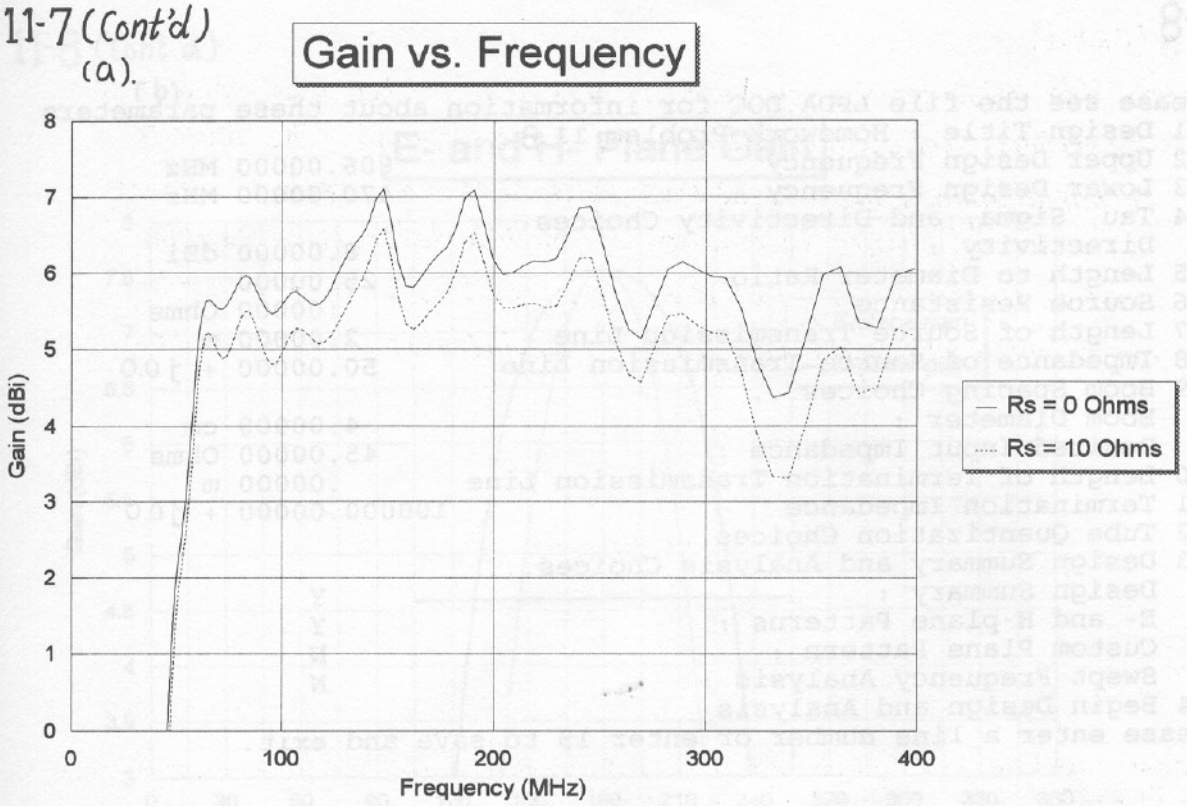
11-7

```
Please see the file LPDA.DOC for information about these parameters
  1 Design Title : Homework Problem 11.7
  2 Upper Design Frequency                  216.00000 MHz
  3 Lower Design Frequency                   54.00000 MHz
  4 Tau, Sigma, and Directivity Choices...
    Directivity :                             7.00000 dBi
  5 Length to Diameter Ratio                 40.00000
  6 Source Resistance                        10.00000 Ohms
  7 Length of Source Transmission Line         .00000 m
  8 Impedance of Source Transmission Line    75.00000 + j0.0
  9 Boom Spacing Choices...
    Boom Diameter :                          10.00000 cm
    Desired Input Impedance :                65.00000 Ohms
 10 Length of Termination Transmission Line     .00000 m
 11 Termination Impedance                    75.00000 + j0.0
 12 Tube Quantization Choices...
 13 Design Summary and Analysis Choices...
    Design Summary :                             Y
    E- and H-plane Patterns :                    N
    Custom Plane Pattern :                       N
    Swept Frequency Analysis :                   Y
 14 Begin Design and Analysis
Please enter a line number or enter 15 to save and exit.
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(Continued)

11-7 (Cont'd)

(a).




(b). A fraction of the source current is absorbed by the source resistance, and a fraction is absorbed by the antenna. The fraction absorbed by the source resistance is $R_s / (R_s + Z_{in})$.

The fraction absorbed by the antenna is $\frac{Z_{in}}{R_s + Z_{in}}$

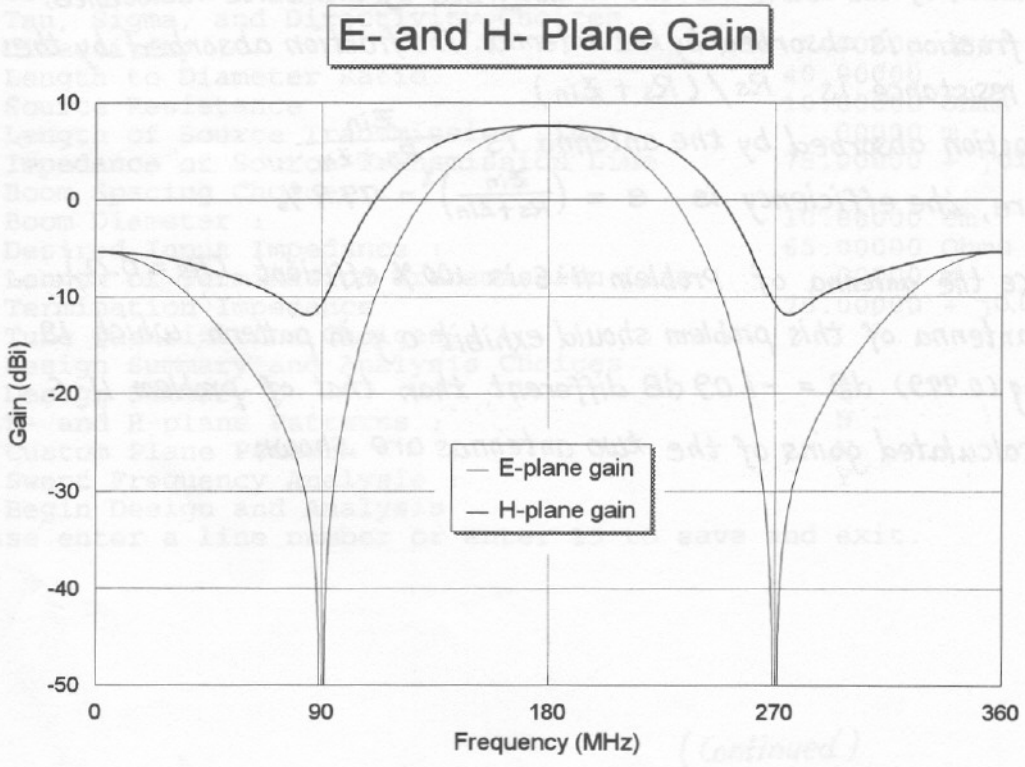
Therefore, the efficiency is $e = \left(\frac{Z_{in}}{R_s + Z_{in}}\right)^2 = 77.9\%$

(c). Since the antenna of Problem 11-6 is 100% efficient ($R_s = 0\,\Omega$), the antenna of this problem should exhibit a gain pattern which is $10 \cdot \log(0.779)$ dB = $-1.09$ dB different than that of problem 11-6. The calculated gains of the two antennas are shown

11-8

```
Please see the file LPDA.DOC for information about these parameters
  1 Design Title : Homework Problem 11.8
  2 Upper Design Frequency                          806.00000 MHz
  3 Lower Design Frequency                          470.00000 MHz
  4 Tau, Sigma, and Directivity Choices...
    Directivity :                                     8.00000 dBi
  5 Length to Diameter Ratio                         25.00000
  6 Source Resistance                                  .00000 Ohms
  7 Length of Source Transmission Line                2.00000 m
  8 Impedance of Source Transmission Line            50.00000 + j0.0
  9 Boom Spacing Choices...
    Boom Diameter :                                   4.00000 cm
    Desired Input Impedance :                        45.00000 Ohms
 10 Length of Termination Transmission Line            .00000 m
 11 Termination Impedance                       100000.00000 + j0.0
 12 Tube Quantization Choices...
 13 Design Summary and Analysis Choices...
    Design Summary :                                      Y
    E- and H-plane Patterns :                             Y
    Custom Plane Pattern :                                N
    Swept Frequency Analysis :                            N
 14 Begin Design and Analysis
Please enter a line number or enter 15 to save and exit.
```

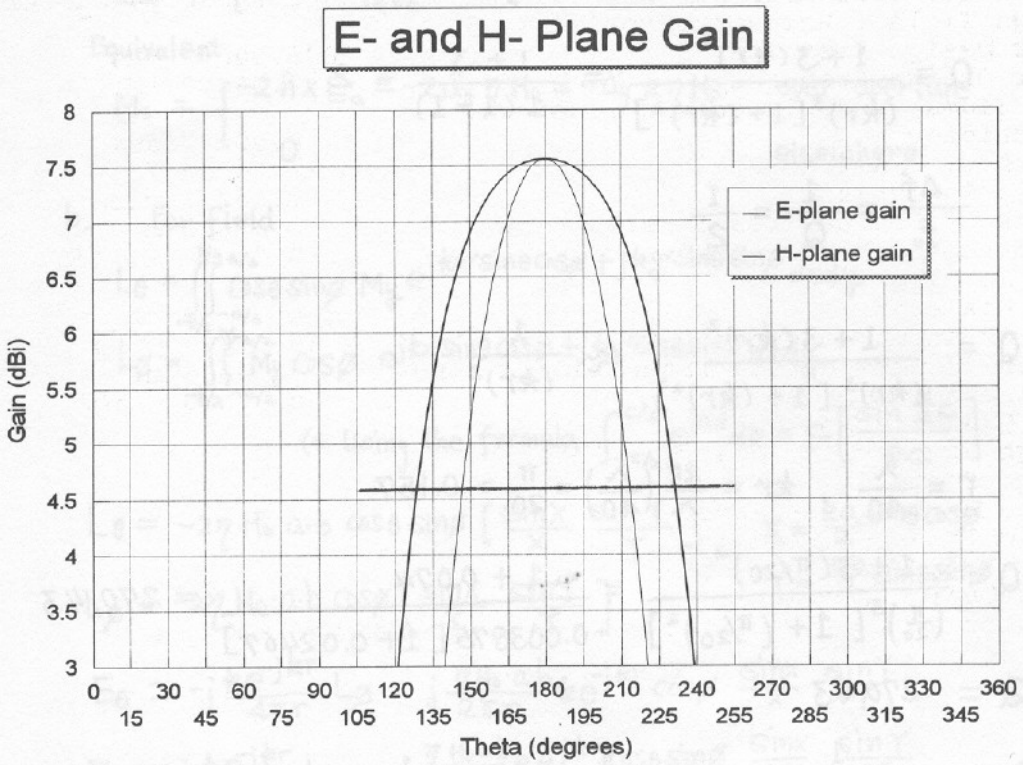
(a).




(Continued)

11-8 (Cont'd)

(b).




3-dB beamwidths are E-plane ~ 67°
H-plane ~ 103°

(C). From the E and H-plane analysis at 600 MHz, the front-to-back ratio is 12.89 dB

(d). The E-plane has deep nulls because this plane corresponds to the plane of the dipoles. The dipole pattern has a null along the axis of the dipole; therefore, the log-periodic dipole array has a null there also.

11-9. $\left.\begin{aligned} 2r &= \lambda/\pi \\ r &= \lambda/\pi \end{aligned}\right\}$ $kr = \left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda}{2\pi}\right) = 1$

a. $Q = \dfrac{1+3(kr)^2}{(kr)^3[1+(kr)^2]} = \dfrac{1+3}{1(1+1)} = 2$

b. $\dfrac{\Delta f}{f_o} = \dfrac{1}{Q} = \dfrac{1}{2}$

11-10. $Q = \dfrac{1+3(kr)^2}{(kr)^3[1+(kr)^2]} \approx \dfrac{1}{(kr)^3}$

a. $r = \dfrac{\lambda}{40}$, $kr = \dfrac{2\pi}{\lambda}\left(\dfrac{\lambda}{40}\right) = \dfrac{\pi}{20} = 0.157$

$Q = \dfrac{1+3(\pi/20)^2}{(\frac{\pi}{20})^3[1+(\pi/20)^2]} = \dfrac{1+0.074}{0.003875[1+0.02467]} = 270.43$

$Q = 270.43$

$Q \simeq \dfrac{1}{\left[\frac{2\pi}{\lambda}\left(\frac{\lambda}{40}\right)\right]^3} = \dfrac{1}{(\pi/20)^3} = \dfrac{1}{0.003875} = 258.01$

b. Larger to better utilize the available volume.

12-1. a. For a uniform plane wave

$$\underline{E}_a = \hat{a}_x \eta H_0 e^{-jkz}\Big|_{z=0} = \hat{a}_x \eta H_0$$

Equivalent

$$\underline{M}_s = \begin{cases} -2\hat{n} \times \underline{E}_a = -2\hat{a}_y \eta H_0 = -\hat{a}_y 2\eta H_0 & \text{over aperture} \\ 0 & \text{elsewhere} \end{cases}$$

b. Far field.

$$L_\theta = \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} \cos\theta \sin\phi \, M_y e^{jkx'\sin\theta\cos\phi + jky'\sin\theta\sin\phi}\, dx'dy'$$

$$L_\phi = \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} M_y \cos\phi \, e^{jkx'\sin\theta\cos\phi + jky'\sin\theta\sin\phi}\, dx'dy'$$

(← Using the formula $\int_{-c/2}^{c/2} e^{j\alpha z} dz = c\left[\frac{\sin\frac{\alpha}{2}c}{\frac{\alpha}{2}c}\right]$

$$L_\theta = -2\eta H_0 \, a\cdot b \cos\theta \sin\phi \left[\frac{\sin X}{X}\frac{\sin Y}{Y}\right], \quad X = \frac{ka}{2}\sin\theta\cos\phi, \quad Y = \frac{kb}{2}\sin\theta\sin\phi$$

$$L_\phi = -2\eta H_0 \cdot a\cdot b \cos\phi \left[\frac{\sin X}{X}\cdot\frac{\sin Y}{Y}\right]$$

$$E_\theta \simeq -j\frac{ke^{-jkr}}{4\pi r} L_\phi = j\frac{\eta H_0 \cdot a\cdot b}{2\pi r} k e^{-jkr}\cos\phi \, \frac{\sin X}{X}\cdot\frac{\sin Y}{Y}$$

$$E_\phi \simeq j\frac{ke^{-jkr}}{4\pi r} L_\theta = -j\frac{\eta H_0 ab}{2\pi r} k e^{-jkr}\cos\theta\sin\phi \, \frac{\sin X}{X}\cdot\frac{\sin Y}{Y}$$

12-2. a. $\underline{H}^i = \hat{a}_x H_0 e^{-jkz}$, $\underline{E}^i = -\hat{a}_y \eta H_0$

$$M_s = \begin{cases} -2\hat{n} \times E^i = -2\cdot\hat{a}_x \cdot \eta H_0 & \text{over aperture} \\ 0 & \text{elsewhere} \end{cases}$$

b. Far field

$$L_\theta = -2\eta H_0 \, a\cdot b \cdot \cos\theta \cdot \cos\phi \cdot \frac{\sin X}{X}\cdot\frac{\sin Y}{Y}$$

$$L_\phi = 2\eta H_0 \cdot a\cdot b \sin\phi \, \frac{\sin X}{X}\cdot\frac{\sin Y}{Y}$$

$$X = \frac{ka}{2}\sin\theta\cos\phi, \quad Y = \frac{kb}{2}\sin\theta\sin\phi$$

$$E_\theta \simeq -j\frac{\eta H_0 \cdot a\cdot b}{2\pi r} k e^{-jkr}\sin\phi \, \frac{\sin X}{X}\cdot\frac{\sin Y}{Y}$$

$$E_\phi \simeq -j\frac{\eta H_0 \cdot a\cdot b}{2\pi r} k e^{-jkr}\cos\theta\cos\phi \, \frac{\sin X}{X}\cdot\frac{\sin Y}{Y}$$

12-3. a. $\underline{E}^i = \hat{a}_y E_o e^{-jkz}$

$$M_s = \begin{cases} -2\hat{n} \times E_o \hat{a}_y = \hat{a}_x 2E_o, & \text{over aperture} \\ 0, & \text{elsewhere} \end{cases}$$

b. Far field

$$L_\theta = 2E_o \cdot a \cdot b \cos\theta \cos\phi \frac{\sin X}{X} \cdot \frac{\sin Y}{Y}, \quad X = \frac{ka}{2}\sin\theta\cos\phi, \; Y = \frac{kb}{2}\sin\theta \cdot \sin\phi$$

$$L_\phi = -2E_o \cdot a \cdot b \cdot \sin\phi \frac{\sin X}{X} \cdot \frac{\sin Y}{Y}.$$

$$E_\theta \simeq -j \frac{ke^{-jkr}}{4\pi r} L_\phi, \quad E_\phi \simeq j \frac{ke^{-jkr}}{4\pi r} L_\theta$$

12-4. This is the same problem as pro.(12-1) except $\underline{E}_a = \hat{a}_x E_o e^{-jkz}$

Equivalent Magnetic current

a. $$M_s = \begin{cases} -2n \times \underline{E}_a = -\hat{a}_y 2E_o, & \text{over aperture} \\ 0, & \text{elsewhere} \end{cases}$$

b. Far field

$$E_\theta \simeq +j \frac{E_o \cdot a \cdot b\, k}{2\pi r} e^{-jkr} \cdot \cos\phi \; \frac{\sin X}{X} \; \frac{\sin Y}{Y}$$

$$E_\phi \simeq -j \frac{E_o a b}{2\pi r} k e^{-jkr} \cos\theta \sin\phi \; \frac{\sin X}{X} \cdot \frac{\sin Y}{Y}$$

$$X = \frac{ka}{2}\sin\theta\cos\phi$$

$$Y = \frac{kb}{2}\sin\theta\sin\phi$$

12-5. Write an expression for the incident field

$$\underline{E} = \hat{a}_z E_o e^{jk(y\sin\phi_o - x\cos\phi_o)}$$

At $x = 0$, $\underline{E} = \hat{a}_z E_o e^{jky'\sin\phi_o}$

Using equivalent

$$\underline{M} = -2\hat{n} \times \underline{E} = \hat{a}_y 2E_o e^{jky'\sin\phi_o}$$

Thus $N_\theta = N_\phi = 0$

$$L_\theta = \iint M_y \cos\theta \sin\phi \, e^{jk(y'\sin\theta\sin\phi + z'\cos\theta)} \, dy'dz'$$

$$= +2E_o \cos\theta \sin\phi \int_{-b/2}^{b/2} e^{jkz'\cos\theta} dz' \int_{-a/2}^{a/2} e^{jky'(\sin\phi_o + \sin\theta\sin\phi)} dy'$$

(Continued)

12-5 (Cont'd)

Using $\int_{-c/2}^{c/2} e^{j\alpha z}\,dz = C\cdot\left(\frac{\sin\frac{\alpha}{2}C}{\frac{\alpha}{2}C}\right)$

$L_\theta = 2E_0\cos\theta\sin\phi\left[b\frac{\sin Z}{Z}\right]\left[a\frac{\sin Y}{Y}\right]$, where $Z=\frac{kb}{2}\cos\theta$, $Y=\frac{ka}{2}(\sin\theta\sin\phi+\sin\phi_0)$

$L_\phi = 2E_0\cos\phi\int_{-b/2}^{b/2} e^{jkz'\cos\theta}\,dz'\int_{-a/2}^{a/2} e^{jk(\sin\theta\sin\phi+\sin\phi_0)}\,dy'$

$L_\phi = 2E_0\cdot a\cdot b\cdot\cos\phi\cdot\frac{\sin Y}{Y}\cdot\frac{\sin Z}{Z}$

Thus $E_\theta = \frac{-jke^{-jkr}}{4\pi r}L_\phi = -j\frac{abE_0\cdot\cos\phi\cdot ke^{-jkr}}{2\pi r}\cdot\frac{\sin Y}{Y}\cdot\frac{\sin Z}{Z}$

$E_\phi = j\frac{ke^{-jkr}}{4\pi r}L_\theta = +j\frac{abE_0\cos\theta\sin\phi\,ke^{-jkr}}{2\pi r}\cdot\frac{\sin Y}{Y}\cdot\frac{\sin Z}{Z}$

$H_\theta = -E_\phi/\eta$, $H_\phi = E_\theta/\eta$

12-6. $\underline{E} = (\hat{a}_y E_0\cos\phi_0 + \hat{a}_x E_0\sin\phi_0)\,e^{jk(y\sin\phi_0 - x\cos\phi_0)}$

$\underline{H} = \hat{a}_z\frac{E_0}{\eta}e^{jk(y\sin\phi_0 - x\cos\phi_0)}$

At $x=0$, $\underline{E} = (\hat{a}_y\cos\phi_0 + \hat{a}_x\sin\phi_0)E_0 e^{jky\sin\phi_0}$

Using Equivalent $\underline{M} = -2\hat{n}\times\underline{E} = -2\hat{a}_z\cdot\cos\phi_0\cdot E_0 e^{jky\sin\phi_0}$, Thus $N_\theta, N_\phi = 0$

$L_\theta = -\iint M_z\sin\theta\, e^{jk(y'\sin\theta\sin\phi + z'\cos\theta)}\,dy'dz'$

$= -2E_0\cdot\sin\theta\cdot\cos\phi_0\cdot\int_{-b/2}^{b/2} e^{jkz'\cos\theta}\,dz'\int_{-a/2}^{a/2} e^{jky'(\sin\phi_0+\sin\theta\sin\phi)}\,dy'$

$L_\theta = -2E_0\sin\theta\cos\phi_0\cdot\left[b\frac{\sin Z}{Z}\right]\left[a\frac{\sin Y}{Y}\right]$, where $Z=\frac{kb}{2}\cos\theta$, $Y=\frac{ka}{2}(\sin\theta\sin\phi+\sin\phi_0)$

$L_\phi = 0$

$E_\phi = j\frac{ke^{-jkr}}{4\pi r}L_\theta = -j\frac{E_0 ab\sin\theta}{2\pi r}\cdot\cos\phi_0\cdot e^{-jkr}\cdot k\cdot\frac{\sin Y}{Y}\cdot\frac{\sin Z}{Z}$

$H_\theta = -\frac{E_\phi}{\eta}$

12-7. a. $\underline{M}_S = 2\underline{E}_a \times \hat{n} = (\hat{a}_y \times \hat{a}_z)\, 2E_0 e^{-jk_0 x' \frac{\sqrt{2}}{2}} = \hat{a}_x\, 2E_0 e^{-jk_0 x' \frac{\sqrt{2}}{2}}$

b. $\underline{F}(\underline{r}) = \frac{\epsilon_0}{4\pi} \iint_{\text{Aperture}} \underline{M}_S(\underline{r}') \frac{e^{-jk_0 R}}{R}\, ds'$

In the far-zone

$$\simeq \frac{\epsilon_0}{4\pi} \frac{e^{-jk_0 r}}{r} \int_{-W/2}^{W/2} \int_{-L/2}^{L/2} (\hat{a}_x 2E_0)\, e^{-jk_0 x' \frac{\sqrt{2}}{2}}\, e^{jk_0 (x'\cos\phi + y'\sin\phi)\sin\theta}\, dx'dy'$$

$$= \hat{a}_x \frac{\epsilon_0 E_0}{2\pi} \frac{e^{-jk_0 r}}{r} \int_{-W/2}^{W/2} e^{jk_0 y' \sin\phi \sin\theta}\, dy' \int_{-L/2}^{L/2} e^{jk_0 x'(\cos\phi\sin\theta - \frac{\sqrt{2}}{2})}\, dx'$$

$$= \hat{a}_x \frac{\epsilon_0 E_0 LW}{2\pi} \frac{e^{-jk_0 r}}{r} \frac{\sin(A)}{A} \frac{\sin(B)}{B}$$

where: $A = \frac{k_0 L}{2}(\cos\phi\sin\theta - \frac{\sqrt{2}}{2})$, $B = \frac{k_0 W}{2}\sin\phi\sin\theta$

$$F_\theta = \hat{a}_\theta \cdot \underline{F} = \frac{\epsilon_0 E_0 LW}{2\pi} \frac{e^{-jk_0 r}}{r} \left\{ \cos\theta\cos\phi \frac{\sin(A)}{A} \cdot \frac{\sin(B)}{B} \right\}$$

$$F_\phi = \hat{a}_\phi \cdot \underline{F} = \frac{-\epsilon_0 E_0 LW}{2\pi} \frac{e^{-jk_0 r}}{r} \left\{ \sin\phi \frac{\sin(A)}{A} \cdot \frac{\sin(B)}{B} \right\}$$

$$E_\theta \simeq -j\omega\eta_0 F_\phi = jE_0 \frac{LW}{\lambda} \frac{e^{-jk_0 r}}{r} \left\{ \sin\phi \cdot \frac{\sin(A)}{A} \cdot \frac{\sin(B)}{B} \right\}$$

$$E_\phi \simeq j\omega\eta_0 F_\theta = -jE_0 \frac{LW}{\lambda} \frac{e^{-jk_0 r}}{r} \left\{ \cos\theta\cos\phi \frac{\sin(A)}{A} \cdot \frac{\sin(B)}{B} \right\}$$

c. Radiation intensity is maximum when $A=B=0$

$B=0 \rightarrow \phi = 0$

$A=0 \rightarrow \sin\theta = \frac{\sqrt{2}}{2} \rightarrow \theta = \frac{\pi}{4} = 45^\circ$

12-8. $\underline{E}_a = \hat{a}_z E_0$, $\underline{M}_S = -2\hat{n} \times \underline{E}_a = -2\hat{a}_x \times \hat{a}_z E_0 = \hat{a}_y 2E_0$ Thus

$M_y = 2E_0$, $M_x = M_z = J_x = J_y = J_z = 0$

$N_\theta = N_\phi = 0$

$$L_\theta = \iint_{Sa} [\overset{0}{\cancel{M_x}} \cos\theta\cos\phi + M_y \cos\theta\sin\phi - \overset{0}{\cancel{M_z}} \sin\theta]\, e^{jk[y'\sin\theta\sin\phi + z'\cos\theta)}\, dy'dz'$$

$$= 2E_0 \cos\theta \cdot \sin\phi \int_{-a/2}^{a/2} e^{jky'\sin\theta\sin\phi}\, dy' \int_{-b/2}^{b/2} e^{jkz'\cos\theta}\, dz'$$

$$L_\theta = 2E_0 ab [\cos\theta\sin\phi \frac{\sin Y}{Y} \frac{\sin Z}{Z}],\ Y = \frac{ka}{2}\sin\theta\cdot\sin\phi,\ Z = \frac{kb}{2}\cos\theta$$

(Continued)

12-8 (Cont'd)

$$L_\phi = \iint_{Sa} [-M_x \sin\phi + M_y \cos\phi]\, e^{jkcy'\sin\theta\cdot\sin\phi + z'\cos\theta)}\, dy'dz'$$

$$= 2E_o \cdot ab\left[\cos\phi \cdot \frac{\sin Y}{Y} \cdot \frac{\sin Z}{Z}\right]$$

Using (12-10a) – (12-10f)

$$E_r \simeq 0$$

$$E_\theta \simeq -\frac{jke^{-jkr}}{4\pi r}[L_\phi + \eta N_\theta] = -j\frac{abkE_o e^{jkr}}{2\pi r}\left[\cos\phi \cdot \frac{\sin Y}{Y}\,\frac{\sin Z}{Z}\right]$$

$$E_\phi \simeq \frac{jke^{-jkr}}{4\pi r}[L_\theta - \eta N_\phi] = -j\frac{abkE_o e^{jkr}}{2\pi r}\left[-\cos\theta\cdot\sin\phi\,\frac{\sin Y}{Y}\cdot\frac{\sin Z}{Z}\right]$$

$$H_r \simeq 0,\quad H_\theta \simeq -\frac{E_\phi}{\eta},\quad H_\phi \simeq +\frac{E_\theta}{\eta}$$

12-9. $\underline{E}_a = \hat{a}_x E_o$, $M_s = -2\hat{n} \times \underline{E}_a = -2\hat{a}_y \times \hat{a}_x E_o = \hat{a}_z 2E_o$

Thus $M_z = 2E_o$, $M_x = M_y = J_x = J_y = J_z = 0$

$$N_\theta = N_\phi = 0$$

$$L_\theta = \iint_{Sa} [M_x \cos\theta\cdot\cos\phi + M_y \cos\theta\sin\phi - M_z\cdot\sin\theta]\, e^{jk(x'\sin\theta\cos\phi + z'\cos\theta)}\, dx'dz'$$

$$= -2E_o \sin\theta \int_{-a/2}^{a/2} e^{jkz'\cos\theta}\, dz' \int_{-b/2}^{b/2} e^{jkx'\sin\theta\cos\phi}\, dx'$$

$$L_\theta = -2E_o ab\left[\sin\theta\cdot\frac{\sin x}{x}\cdot\frac{\sin z}{z}\right],\quad x = \frac{kb}{2}\sin\theta\cos\phi,\quad z = \frac{ka}{2}\cos\theta$$

$$L_\phi = \iint_{Sa} [-M_x \sin\phi + M_y \cos\phi]\, e^{jk(x'\sin\theta\cos\phi + z'\cos\theta)}\, dx'dz' = 0$$

Using (12-10a) – (12-10f)

$$E_r \simeq 0,\quad E_\theta \simeq -\frac{jke^{-jkr}}{4\pi r}[L_\phi + \eta N_\theta] = 0$$

$$E_\phi \simeq \frac{jke^{-jkr}}{4\pi r}[L_\theta - \eta N_\phi] = -j\frac{abkE_o e^{jkr}}{2\pi r}\left[\sin\theta\cdot\frac{\sin x}{x}\cdot\frac{\sin z}{z}\right]$$

$$H_r \simeq 0,\quad H_\theta \simeq -\frac{E_\phi}{\eta},\quad H_\phi \simeq \frac{E_\theta}{\eta} = 0$$

12-10. The only difference between this problem and 12-8 is that for the y variations the integral reduces to

$$\int_{-a/2}^{a/2} \cos\left(\frac{\pi}{a}y'\right) e^{jky'\sin\theta\sin\phi}\, dy' = -\left(\frac{\pi a}{2}\right)\frac{\cos\left(\frac{ka}{2}\sin\theta\sin\phi\right)}{\left(\frac{ka}{2}\sin\theta\sin\phi\right)^2 - \left(\frac{\pi}{2}\right)^2}$$

Thus $E_r \simeq 0$

$$E_\theta \simeq -j\frac{ke^{-jkr}}{4\pi r}L_\phi = j\frac{abkE_o e^{-jkr}}{4r}\left[\cos\phi\cdot\frac{\cos Y}{(Y)^2 - (\pi/2)^2}\cdot\frac{\sin Z}{Z}\right],\quad Y = \frac{ka}{2}\sin\theta\sin\phi$$

$$E_\phi \simeq j\frac{ke^{-jkr}}{4\pi r}L_\theta = +j\frac{abkE_o e^{jkr}}{4r}\left[-\cos\theta\sin\phi\,\frac{\cos Y}{(Y)^2 - (\pi/2)^2}\cdot\frac{\sin Z}{Z}\right],\quad Z = \frac{kb}{2}\cos\theta$$

$$H_r \simeq 0,\quad H_\theta \simeq -E_\phi/\eta,\quad H_\phi \simeq E_\theta/\eta$$

12-11. The only difference between this problem and 12.9 is that for the $z$ variations the integral reduces to

$$\int_{-a/2}^{a/2} \cos\left(\frac{\pi}{a}z'\right) e^{jkz'\cos\theta}\, dz' = -\left(\frac{\pi a}{2}\right) \frac{\cos\left(\frac{ka}{2}\cos\theta\right)}{\left(\frac{ka}{2}\cos\theta\right)^2 - \left(\frac{\pi}{2}\right)^2}$$

Thus $E_r \simeq 0$

$E_\theta \simeq 0$

$$E_\phi \simeq j\frac{k e^{-jkr}}{4\pi r} L_\theta = j\frac{ab k E_0 e^{-jkr}}{4r}\left[\sin\theta \frac{\sin X}{X} \frac{\cos Z}{(Z)^2 - (\frac{\pi}{2})^2}\right]$$

$X = \frac{kb}{2}\sin\theta\cos\phi$, $Z = \frac{ka}{2}\cos\theta$

$H_r \simeq 0$, $H_\theta \simeq -E_\phi/\eta$, $H_\phi \simeq E_\theta/\eta = 0$

12-12. a. $\underline{E}_a = \hat{a}_z E_0$, $\underline{M}_s = -\hat{n} \times \underline{E}_a = \hat{a}_y E_0 \Rightarrow M_x = M_z = 0$, $M_y = E_0$

$\underline{H}_a = -\hat{a}_y \frac{E_0}{\eta}$, $\underline{J}_s = \hat{n} \times \underline{H}_a = \hat{a}_x \times (-\hat{a}_y \frac{E_0}{\eta}) = -\hat{a}_z \frac{E_0}{\eta}$

$\Rightarrow J_x = J_y = 0$, $J_z = -E_0/\eta$

From Prob. 12.8.

$L_\theta = E_0 \cdot a \cdot b\left[\cos\theta \sin\phi \cdot \frac{\sin Y}{Y} \cdot \frac{\sin Z}{Z}\right]$, $Y = \frac{ka}{2}\sin\theta\sin\phi$, $Z = \frac{kb}{2}\cos\theta$

$L_\phi = E_0 \cdot ab\left[\cos\phi \cdot \frac{\sin Y}{Y} \cdot \frac{\sin Z}{Z}\right]$

$$N_\theta = \iint_{Sa} \left[\cancel{J_x}^{\,0} \cos\theta\cos\phi + \cancel{J_y}^{\,0} \cos\theta\sin\phi - J_z \sin\theta\right] e^{jk(y'\sin\theta\sin\phi + z'\cos\theta)}\, dy'dz'$$

$$= + \frac{E_0 ab}{\eta}\left[\sin\theta \cdot \frac{\sin Y}{Y} \cdot \frac{\sin Z}{Z}\right]$$

$$N_\phi = \iint_{Sa} \left[-\cancel{J_x}^{\,0} \sin\phi + \cancel{J_y}^{\,0} \cos\phi\right] e^{jk(y'\sin\theta\sin\phi + z'\cos\theta)}\, dy'dz' = 0$$

$E_r \simeq 0$

$$E_\theta \simeq -j\frac{k e^{-jkr}}{4\pi r}\left[L_\phi + \eta N_\theta\right] = -j\frac{kab E_0 e^{-jkr}}{4\pi r}\left[(\cos\phi + \sin\theta)\frac{\sin Y}{Y} \cdot \frac{\sin Z}{Z}\right]$$

$$E_\phi \simeq j\frac{k e^{-jkr}}{4\pi r}\left[L_\theta - \eta \cancel{N_\phi}^{\,0}\right] = -j\frac{kab E_0 e^{-jkr}}{4\pi r}\left[-\cos\theta\sin\phi \frac{\sin Y}{Y} \cdot \frac{\sin Z}{Z}\right]$$

$H_r \simeq 0$

$H_\theta \simeq -E_\phi/\eta$

$H_\phi \simeq E_\theta/\eta$

b. $E_a = \hat{a}_x E_0$, $\underline{M}_s = -\hat{n} \times \underline{E}_a = \hat{a}_z E_0 \Rightarrow M_x = M_y = 0$, $M_z = E_0$

$\underline{H}_a = -\hat{a}_z \frac{E_0}{\eta}$, $\underline{J}_s = \hat{n} \times \underline{H}_a = \hat{a}_y \times (-\hat{a}_z \frac{E_0}{\eta}) = -\hat{a}_x \frac{E_0}{\eta} \Rightarrow J_y = J_z = 0$, $J_x = -\frac{E_0}{\eta}$

From Problem 12.9

$L_\theta = -E_0 ab\left[\sin\theta \cdot \frac{\sin X}{X} \cdot \frac{\sin Z}{Z}\right]$ (Continued)

12-12 (Cont'd) $L_\phi = 0$

$$x = \frac{kb}{2}\sin\theta\cos\phi\,, \quad z = \frac{ka}{2}\cos\theta$$

$$N_\theta = \iint_{Sa} [J_x\cos\theta\cos\phi + \cancel{J_y\cos\theta\sin\phi}^{\,0} - \cancel{J_z\sin\theta}^{\,0}]\, e^{jk(x'\sin\theta\cos\phi + z'\cos\theta)}\, dx'dz'$$

$$= -\frac{E_0}{\eta}\, ab\left[\cos\theta\cos\phi \cdot \frac{\sin x}{x} \cdot \frac{\sin z}{z}\right]$$

$$N_\phi = \iint_{Sa} [-J_x\sin\phi + J_y\cos\phi]\, e^{jk(x'\sin\theta\cos\phi + z'\cos\theta)}\, dx'dz'$$

$$= \frac{E_0}{\eta}\, ab\left[\sin\phi \cdot \frac{\sin x}{x} \cdot \frac{\sin z}{z}\right]$$

$E_r \simeq 0$

$$E_\theta \simeq -j\frac{ke^{-jkr}}{4\pi r}[\cancel{L_\phi}^{\,0} + \eta N_\theta] = j\frac{kab E_0 e^{-jkr}}{4\pi r}\left[\cos\theta\cos\phi\,\frac{\sin x}{x}\,\frac{\sin z}{z}\right]$$

$$E_\phi \simeq j\frac{ke^{-jkr}}{4\pi r}[L_\theta - \eta N_\phi] = j\frac{kab E_0 e^{-jkr}}{4\pi r}\left[-(\sin\theta + \sin\phi)\cdot\frac{\sin x}{x}\cdot\frac{\sin z}{z}\right]$$

$H_r \simeq 0$, $H_\theta \simeq -E_\phi/\eta$, $H_\phi \simeq E_\theta/\eta$

C. Using a comparison between the fields of Prob 12.8 and 12.12(a), we can write by referring to the fields of Problem 12.10 that

$E_r \simeq 0$

$$E_\theta \simeq j\frac{abkE_0 e^{-jkr}}{8r}\left[(\cos\phi + \sin\theta)\,\frac{\cos Y}{(Y)^2 - (\pi/2)^2}\,\frac{\sin z}{z}\right]$$

$$E_\phi \simeq j\frac{abkE_0 e^{-jkr}}{8r}\left[-\cos\theta\cdot\sin\phi\cdot\frac{\cos Y}{(Y)^2 - (\frac{\pi}{2})^2}\right]$$

$Y = \frac{ka}{2}\sin\theta\sin\phi$, $z = \frac{kb}{2}\cos\theta$, $H_r \simeq 0$, $H_\theta \simeq -E_\phi/\eta$, $H_\phi \simeq E_\theta/\eta$

d. Using a comparison between the fields of Prob 12.9 and 12.12(b) we can write by referring to the fields of Problem 12.11 that

$E_r \simeq 0$

$$E_\theta \simeq j\frac{kabE_0 e^{-jkr}}{8r}\left[\cos\theta\cos\phi\cdot\frac{\sin x}{x}\cdot\frac{\cos z}{(z)^2 - (\pi/2)^2}\right]$$

$$E_\phi \simeq j\frac{kabE_0 e^{-jkr}}{8r}\left[-(\sin\theta + \sin\phi)\cdot\frac{\sin x}{x}\cdot\frac{\cos z}{(z)^2 - (\pi/2)^2}\right]$$

$x = \frac{kb}{2}\sin\theta\cdot\cos\phi$, $z = \frac{ka}{2}\cos\theta$

$H_r \simeq 0$,

$H_\theta \simeq -E_\phi/\eta$

$H_\phi \simeq E_\theta/\eta$

12-13 $\underline{E}_a = \hat{a}_y E_0 \cos(\frac{\pi}{a}x') \Rightarrow \underline{M}_s = -\hat{n} \times \underline{E}_a = \hat{a}_x E_0 \cos(\frac{\pi}{a}x')$

Thus $M_x = E_0 \cos(\frac{\pi}{a}x')$, $M_y = M_z = 0$

$$L_\theta = \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} M_x \cos\theta\cos\phi\, e^{jk(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)}dx'dy'$$

$$= E_0\cos\theta\cos\phi \int_{-a/2}^{a/2}\cos(\tfrac{\pi}{a}x')e^{jkx'\sin\theta\cos\phi}dx' \int_{-b/2}^{b/2} e^{jky'\sin\theta\sin\phi}dy'$$

Since $\int_{-a/2}^{a/2}\cos(\frac{\pi}{a}x')e^{jkx'\sin\theta\cos\phi}dx' = -(\frac{\pi a}{2})\frac{\cos(X)}{(X)^2-(\frac{\pi}{2})^2}$

$\int_{-b/2}^{b/2} e^{jky'\sin\theta\sin\phi}dy' = b\,\frac{\sin Y}{Y}$ , $X = \frac{ka}{2}\sin\theta\cos\phi$, $Y = \frac{kb}{2}\sin\theta\sin\phi$

then $L_\theta = -\frac{\pi ab}{2}E_0\left[\cos\theta\cdot\cos\phi\,\frac{\cos X}{(X)^2-(\frac{\pi}{2})^2}\cdot\frac{\sin Y}{Y}\right]$

Similarly

$$L_\phi = \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} -M_x\sin\phi\, e^{jk(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)}dx'dy'$$

$$= -E_0\sin\phi\int_{-a/2}^{a/2}\cos(\tfrac{\pi}{a}x')e^{jkx'\sin\theta\cos\phi}dx'\int_{-b/2}^{b/2}e^{jky'\sin\theta\sin\phi}dy'$$

$$L_\phi = +\frac{\pi ab}{2}E_0\left[\sin\phi\,\frac{\cos X}{(X)^2-(\frac{\pi}{2})^2}\cdot\frac{\sin Y}{Y}\right]$$

$\underline{H}_a \simeq -\hat{a}_x\frac{E_a}{\eta} \Rightarrow \underline{J}_s = \hat{n}\times\underline{H}_a = \hat{a}_z\times(-\hat{a}_x\frac{E_a}{\eta}) = -\hat{a}_y\frac{E_a}{\eta} = -\hat{a}_y\frac{E_0}{\eta}\cos(\frac{\pi}{a}x')$

$J_x = J_z = 0$, $J_y = -\frac{E_0}{\eta}\cos(\frac{\pi}{a}x')$

$$N_\theta = \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} J_y\cos\theta\sin\phi\, e^{jk(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)}dx'dy'$$

$$= -\frac{E_0}{\eta}\cos\theta\sin\phi\int_{-a/2}^{a/2}\cos(\tfrac{\pi}{a}x')e^{jkx'\sin\theta\cos\phi}dx'\int_{-b/2}^{b/2}e^{jky'\sin\theta\sin\phi}dy'$$

$$= +\frac{\pi abE_0}{2\eta}\left[\cos\theta\sin\phi\cdot\frac{\cos X}{(X)^2-(\pi/2)^2}\cdot\frac{\sin Y}{Y}\right]$$

$$N_\phi = \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} J_y\cos\phi\, e^{jk(x'\cos\phi\sin\theta + y'\sin\theta\sin\phi)}dx'dy'$$

$$= +\frac{\pi ab\,E_0}{2\eta}\left[\cos\phi\cdot\frac{\cos X}{(X)^2-(\frac{\pi}{2})^2}\cdot\frac{\sin Y}{Y}\right]$$

(Continued)

12-13 (Cont'd) Using (12-10a)-(12-10f), we can write that

$$E_r \simeq 0$$

$$E_\theta \simeq -j\frac{ke^{-jkr}}{4\pi r}[L_\phi + \eta N_\theta] = -j\frac{abkE_o e^{-jkr}}{8r}\left[\sin\phi(1+\cos\theta)\frac{\cos X}{(X)^2-(\frac{\pi}{2})^2}\cdot\frac{\sin Y}{Y}\right]$$

$$E_\phi \simeq j\frac{ke^{-jkr}}{4\pi r}[L_\theta - \eta N_\phi] = -j\frac{abkE_o e^{-jkr}}{8r}\left[\cos\phi(1+\cos\theta)\frac{\cos X}{(X)^2-(\frac{\pi}{2})^2}\frac{\sin Y}{Y}\right]$$

$$H_r \simeq 0,\quad H_\theta = -E_\phi/\eta,\quad H_\phi = E_\theta/\eta$$

12-14. $a = 4\lambda,\ b = 3\lambda$

a. From Appendix I

$$\frac{kb}{2}\sin\theta_s \simeq 7.7 \Rightarrow \theta_s = \sin^{-1}\left[\frac{2(7.7)}{kb}\right] = \sin^{-1}\left[\frac{2(7.7)}{2\pi(3)}\right] = 54.785^\circ$$

$$\text{HP}_s = 2\theta_s = 2(54.785) = 109.57^\circ$$

b. From Appendix I, at $\frac{kb}{2}\sin\theta_s = 7.7 \Rightarrow E_\theta = 0.12833$

or $E_\theta = -17.83\text{dB}$

c. $\text{HP}_h(\text{E-plane}) = 114.6\sin^{-1}(\frac{0.443}{3}) = 16.98^\circ$

$\text{HP}_h(\text{H-plane}) \simeq 114.6\sin^{-1}(\frac{0.443}{4}) = 12.72^\circ$

$$D_o \simeq \frac{41{,}253}{16.98(12.72)} = 191 = 22.81\text{dB}$$

From Table 12.1, $D_o = 10.2(\frac{ab}{\lambda^2}) = 10.2(3)(4) = 122.4 = 20.88\text{dB}$

12-15. $a = b = 3\lambda$

Using (12-37), $D_o = \frac{4\pi}{\lambda^2}ab = 4\pi(3)^2 = 113.1 = 20.53\text{dB}$

Using the computer program at the end of Chapter 2.

$D_o = 119.46 = 20.77\text{dB}$

12-16. $a = b = 3\lambda$

Using (12-37), $D_o = \frac{4\pi}{\lambda^2}ab = 4\pi(3)^2 = 113.1 = 20.53\text{dB}$

Using the computer program at the end of Chapter 2.

$D_o = 119.38 = 20.77\text{dB}$

12-17. Using the computer program at the end of Chapter 2.

a. $a = 3\lambda, b = 2\lambda$; $D_o = 62.437 = 17.95\text{dB}$

b. $a = b = 3\lambda$; $D_o = 93.174 = 19.69\text{dB}$

Using Table 12.1

a. $a = 3\lambda, b = 2\lambda$; $D_o = 0.81(4\pi)(3)(2) = 61.07 = 17.85\text{dB}$

b. $a = b = 3\lambda$; $D_o = 0.81(4\pi)(3)(3) = 91.61 = 19.62\text{dB}$

12-18 Using the computer program at the end of Chapter 2

a. $a=3\lambda, b=2\lambda$ ; $D_o = 63.961 = 18.06$ dB

b. $a=b=3\lambda$ ; $D_o = 94.306 = 19.75$dB

12-19 $a=3\lambda, b=2\lambda$

a. $\Theta_h$ (E-plane) $= 50.6/2 = 25.30°$

b. $\Theta_h$ (H-plane) $= 68.8/3 = 22.93°$

c. $\Theta_n$ (E-plane) $= 114.6/2 = 57.30°$

d. $\Theta_n$ (H-plane) $= 171.9/3 = 57.30°$

e. $E_\theta$ (E-plane) $= -13.26$dB

f. $E_\phi$ (H-plane) $= -23$dB

Using the data from Figure 12.13 and 12.14

a. $\Theta_h$ (E-plane) $= 25.6°$

b. $\Theta_h$ (H-plane) $= 21°$

c. $\Theta_n$ (E-plane) $= 60.0°$

d. $\Theta_n$ (H-plane) $= 60°$

e. $E_\theta$ (E-plane) $= -13.26$ dB

f. $E_\phi$ (H-plane) $= -26$ dB

12-20 From Figure 12.15, for a 90% efficiency

$$u = \frac{ka}{2}\sin\theta_1 \simeq 3.18 = \frac{kb}{2}\sin\theta_1$$

For $\theta_1 = 37/2 = 18.5°$

$$a=b=\frac{2(3.18)}{k\sin(18.5°)} = 3.19\lambda$$

12-21 $\underline{E}_a = \hat{a}_y E_o \cos(\frac{\pi}{a}x')$, $\underline{H}_a \simeq -\hat{a}_x \frac{E_o}{\eta}\cos(\frac{\pi}{a}x')$

$$\underline{W}_{ave} = \frac{1}{2}\mathrm{Re}(\underline{E}\times\underline{H}^*) = \hat{a}_z \frac{|E_o|^2}{2\eta}\cos^2(\frac{\pi}{a}x')$$

$$P_{rad} = \frac{1}{2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}\frac{|E_o|^2}{\eta}\cos^2(\frac{\pi}{a}x')dx'dy' = \frac{|E_o|^2 b}{2\eta}\int_{-a/2}^{a/2}\left[\frac{1+\cos(\frac{2\pi}{a}x')}{2}\right]dx'$$

$$P_{rad} = \frac{|E_o|^2 b}{4\eta}\left[x' + \frac{a}{2\pi}\sin(\frac{2\pi}{a}x')\right]_{-a/2}^{a/2} = \frac{ab|E_o|^2}{4\eta}$$

From Table 12.1, at $\theta = 0°$

$$E_\theta|_{max} = j\frac{2}{\pi}\frac{abkE_o e^{-jkr}}{2\pi r}\sin\phi,\quad E_\phi|_{max} = j\frac{2}{\pi}\frac{abkE_o e^{-jkr}}{2\pi r}\cos\phi = j\frac{2abe^{-jkr}}{\pi\lambda r}\cos\phi$$

$$|E|_{max} = \sqrt{|E_\theta|^2_{max} + |E_\phi|^2_{max}} = |E_o|\frac{4}{r^2}(\frac{ab}{\pi\lambda})^2 \Rightarrow U_{max} = \frac{r^2}{2\eta}|E|^2_{max} = \frac{2(ab)^2|E_o|^2}{(\pi\lambda)^2\eta}$$

$$D_o = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi[2(ab)^2|E_o|^2]/[(\pi\lambda)^2\eta]}{ab|E_o|^2/4\eta} = \frac{8}{\pi^2}[ab(\frac{4\pi}{\lambda^2})]$$

$$A_{em} = \frac{\lambda^2}{4\pi}D_o = \frac{8}{\pi^2}(ab) = \frac{8}{\pi^2}A_p =: \epsilon_{ap}A_p$$

12-22. $D_o = \frac{4\pi}{\lambda^2} A_{em} = \frac{4\pi}{\lambda^2} \varepsilon_{ap} A_p = \varepsilon_{ap} \frac{4\pi}{\lambda^2}(ab)$

$D_o = \varepsilon_{ap} \frac{4\pi(ab)}{\lambda^2} = \varepsilon_{ap} \frac{4\pi}{\lambda^2}\left(\frac{\lambda^2}{8}\right) = \varepsilon_{ap} \frac{\pi}{2}$

a. Triangular : $\varepsilon_{ap} = 75\% = 3/4$

$D_o = \frac{3}{4}\left(\frac{\pi}{2}\right) = 1.1781 = 0.7118\ dB$

$D_o = 1.1781 = 0.7118\ dB$

b. Cosine Square ; $\varepsilon_{ap} = 66.67\% = 2/3$

$D_o = \varepsilon_{ap} \frac{\pi}{2} = \frac{2}{3}\left(\frac{\pi}{2}\right) = \frac{\pi}{3} = 1.0472 = 0.2\ dB$

$D_o = 1.0472 = 0.2\ dB$

12-23. $D_o = 23\ dB = 10^{2.3} = 199.526$

$e_t = e_{cd} e_r = 0.9(1) = 0.9$

$G_{To} = e_t D_o = 0.9(199.526) = 179.5736 = 22.54\ dB$

$A_{em} = \frac{\lambda^2}{4\pi} e_t D_o = \frac{\lambda^2}{4\pi} G_o$

$A_{em} = \frac{\lambda^2}{4\pi}(179.5736) = \frac{3^2}{4\pi} \cdot 179.5736 = 128.61\ cm^2$

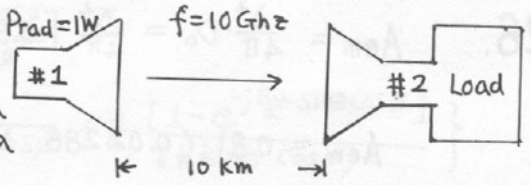
$\lambda = \frac{3 \times 10^8}{10 \times 10^9} = 0.03\ m = 3\ cm$

$\varepsilon_{ap} = \frac{A_{em}}{A_p} = \frac{128.61}{200} = 0.6431 = 64.31\%$

12-24. a. $\lambda = \frac{3 \times 10^{10}\ cm/s}{10 \times 10^9\ Hz} = 3\ cm$

$a = 0.9'' = 2.286\ cm = 0.762\lambda$

$b = 0.4'' = 1.016\ cm = 0.339\lambda$




Power density for isotropic source;

$W_o = \frac{P_{rad}}{4\pi R^2} = \frac{1\ Watt}{4\pi(10 \times 10^3)^2} = 7.96 \times 10^{-10}\ W/m^2$

Directivity from Table 12.1, 12.2.

$D_o = \frac{8}{\pi^2}\left[\frac{4\pi}{\lambda^2} ab\right] = \frac{32}{\pi}(0.762)(0.339) = 2.63$

Incident Power density

$W_i = W_o D_o = (7.96 \times 10^{-10}\ W/m^2)(2.63) \Rightarrow W_i = 2.09 \times 10^{-9}\ W/m^2$

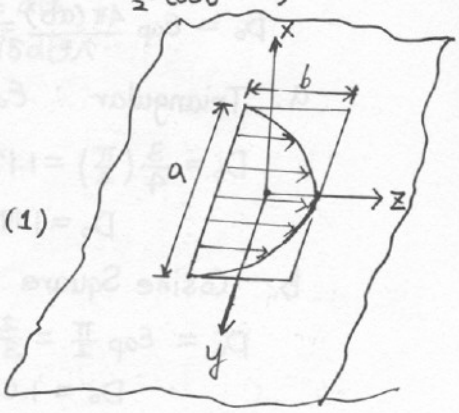
b. The maximum power that can be delivered to a matched load.

$A_{em} = \varepsilon_{ap} A_p = 0.81\ ab = 1.88 \times 10^{-4}\ m^2$

$P_{max} = W_i A_{em} = (2.09 \times 10^{-9}\ W/m^2)(1.88 \times 10^{-4}\ m^2) = 3.94 \times 10^{-13}\ W$

$P_{max} = 3.94 \times 10^{-13}\ W$

12-25 $\underline{E} = \hat{a}_\theta\, j\frac{\omega\mu b I_o e^{-jkr}}{4\pi r}\frac{\sin(\frac{kb}{2}\cos\theta)}{\frac{kb}{2}\cos\theta} = -j\frac{\omega\mu I_o e^{-jkr}}{4\pi r}\left\{-\hat{a}_\theta\, b\frac{\sin(\frac{kb}{2}\cos\theta)}{\frac{kb}{2}\cos\theta}\right\}$




a. $\underline{E} = -j\frac{\omega\mu I_o e^{-jkr}}{4\pi r}\cdot \underline{h}_e(\theta)$

$\underline{h}_e(\theta) = -\hat{a}_\theta\, b\frac{\sin(\frac{kb}{2}\cos\theta)}{\frac{kb}{2}\cos\theta}$

b. $\underline{h}_e(\theta)\big|_{max} = -\hat{a}_\theta\, b\frac{\sin(\frac{kb}{2}\cos\theta)}{\frac{kb}{2}\cos\theta}\Big|_{max} = -\hat{a}_\theta\, b\,(1)$

when $\frac{kb}{2}\cos\theta = 0 \Rightarrow \theta = 90°$

C. $p = \frac{|\underline{h}_e(\theta)\cdot\underline{E}^{inc}|^2}{|\underline{h}_e(\theta)|^2|\underline{E}^{inc}|^2}$

$\underline{E}^{inc} = \hat{a}_\theta\, j\eta\frac{kI_o l e^{-jkr}}{8\pi r}\sin\theta$

$|\underline{h}_e(\theta)\,\underline{E}^{inc}|^2 = \left|(-\hat{a}_\theta b)\cdot\left[\hat{a}_\theta\, j\eta\frac{kI_o l e^{-jkr}}{8\pi r}\sin\theta\right]_{\theta=\frac{\pi}{2}}\right|^2 = \left|-j\eta b\frac{kI_o l e^{-jkr}}{8\pi r}\right|^2 = \left|\eta\frac{bkI_o l}{8\pi r}\right|^2$

$|h_e(\theta)|^2 = |b|^2$

$|\underline{E}^{inc}|^2 = \left|j\eta\frac{kI_o l e^{-jkr}}{8\pi r}\sin\theta\Big|_{\theta=\frac{\pi}{2}}\right|^2 = \left|\eta\frac{kI_o l}{8\pi r}\right|^2$

$p = \frac{\left|\eta\frac{kbI_o l}{8\pi r}\right|^2}{|b|^2\left|\eta\frac{kI_o l}{8\pi r}\right|^2} = 1 = 0\,dB$

12-26. $A_{em} = \frac{\lambda^2}{4\pi}\cdot D_o = \frac{\lambda^2}{4\pi}\left[0.81\cdot a\cdot b\cdot\left(\frac{4\pi}{\lambda^2}\right)\right] = 0.81\,ab$

$A_{em} = 0.81\cdot(0.02286 \times 0.0106) = 0.81\cdot(2.32257\times10^{-4})$

$= 1.88\times10^{-4}\ m^2$

The maximum power that can be delivered to matched load

$P_{max} = W_i\cdot A_{em} = (10^{-4}\ watts/m^2)\cdot(1.88\times10^{-4}\ m^2)$

$= 1.88\times10^{-8}\ watts. = 0.0188\ \mu Watts.$

12-27.

a. $\underline{E}_a = \begin{cases} \hat{a}_y\, 2E_o\left(\frac{2}{a}x'+1\right), & -\frac{a}{2} \le x' \le 0 \\ \hat{a}_y\, 2E_o\left(-\frac{2}{a}x'+1\right), & 0 \le x' \le \frac{a}{2} \end{cases}$

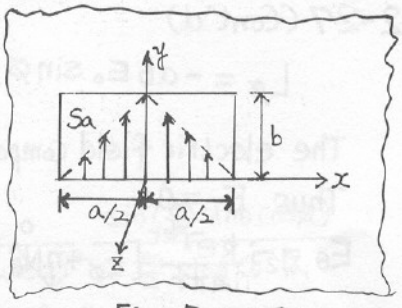


Fig. P12-27(a)

$\underline{M}_s = -2\hat{n} \times \underline{E}_a = -2\hat{a}_z \times \hat{a}_y E_y = \hat{a}_x 2E_y$

$M_y = M_z = J_x = J_y = J_z = 0$

$M_x = \begin{cases} 2E_o\left(\frac{2}{a}x'+1\right), & -\frac{a}{2} \le x' \le 0 \\ 2E_o\left(-\frac{2}{a}x'+1\right), & 0 \le x' \le \frac{a}{2} \end{cases}$

Using (12-12a)-(12-12d)

$N_\theta = N_\phi = 0$

$$L_\theta = \iint_S [M_x \cos\theta\cos\phi + \cancel{M_y}^{\,0}\cos\theta\sin\phi - \cancel{M_z}^{\,0}\sin\theta]\, e^{jkr'\cos\psi}\, ds'$$

$$L_\theta = 2E_o\cos\theta\cos\phi \int_0^b \left\{ \int_{-a/2}^{0} \left(\frac{2}{a}x'+1\right) e^{jkx'\sin\theta\cos\phi}\, dx' + \int_0^{a/2} \left(-\frac{2}{a}x'+1\right) e^{jkx'\sin\theta\cos\phi}\, dx' \right\} e^{jky'\sin\theta\sin\phi}\, dy'$$

$$\int_0^b e^{jky'\sin\theta\sin\phi}\, dy' = \frac{e^{jky'\sin\theta\sin\phi}}{jk\sin\theta\cdot\sin\phi} = e^{-j\frac{kb}{2}\sin\theta\sin\phi} \cdot b \cdot \frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\frac{kb}{2}\sin\theta\sin\phi}$$

$$\int_{-a/2}^{0} \left(\frac{2}{a}x'+1\right) e^{jkx'\sin\theta\cos\phi}\, dx' = \left[\left(\frac{2}{a}x'+1\right)\frac{e^{jkx'\sin\theta\cos\phi}}{jk\sin\theta\cos\phi}\right]_{-a/2}^{0} - \frac{2}{a}\int_{-a/2}^{0} \frac{e^{jkx'\sin\theta\cos\phi}}{jk\sin\theta\cos\phi}\, dx'$$

$u = \frac{2}{a}x'+1 \qquad dv = e^{jkx'\sin\theta\cos\phi}\, dx'$

$du = \frac{2}{a}\, dx' \qquad v = \frac{e^{jkx'\sin\theta\cos\phi}}{jk\sin\theta\cos\phi}$

$$\int_{-a/2}^{0} \left(\frac{2}{a}x'+1\right) e^{jkx'\sin\theta\cos\phi}\, dx' = \left\{ \frac{1}{jk\sin\theta\cos\phi} + \frac{2}{a}\frac{\left[1-e^{-j\frac{ka}{2}\sin\theta\cos\phi}\right]}{(k\sin\theta\cos\phi)^2} \right\}$$

Similarly

$$\int_0^{a/2} \left(-\frac{2}{a}x'+1\right) e^{jkx'\sin\theta\cos\phi}\, dx' = \left\{ \frac{-1}{jk\sin\theta\cos\phi} + \frac{2}{a}\frac{\left[1-e^{j\frac{ka}{2}\sin\theta\cos\phi}\right]}{(k\sin\theta\cos\phi)^2} \right\}$$

Combining terms, we can write

$$L_\theta = 2E_o\cos\theta\cos\phi \left\{ b\, e^{-j\Upsilon}\frac{\sin\Upsilon}{\Upsilon} \right\} \frac{2}{a}\frac{2\left[1-\cos\left(\frac{ka}{2}\sin\theta\cos\phi\right)\right]}{(k\sin\theta\cos\phi)^2}$$

$$= 8E_o\left(\frac{b}{a}\right)\cos\theta\cos\phi \cdot e^{-j\Upsilon}\frac{\sin\Upsilon}{\Upsilon} \cdot \frac{2\sin^2\left(\frac{ka}{4}\sin\theta\cos\phi\right)}{(k\sin\theta\cos\phi)^2}$$

$$L_\theta = abE_o\cos\theta\cos\phi\, e^{-j\Upsilon}\frac{\sin\Upsilon}{\Upsilon} \cdot \frac{\sin^2\left(\frac{ka}{4}\sin\theta\cos\phi\right)}{\left(\frac{ka}{4}\sin\theta\cos\phi\right)^2}, \quad \Upsilon = \frac{kb}{2}\sin\theta\sin\phi$$

Similarly

$$L_\phi = \iint_{Sa} [-M_x\sin\phi + \cancel{M_y}^{\,0}\cos\phi]\, e^{jkr'\cos\psi}\, ds'$$

(Continued)

12-27 (Cont'd)

$$L_\phi = -ab E_o \sin\phi\, e^{j\Upsilon} \frac{\sin\Upsilon}{\Upsilon} \cdot \frac{\sin^2\left(\frac{ka}{4}\sin\theta\cos\phi\right)}{\left(\frac{ka}{4}\sin\theta\cdot\cos\phi\right)^2}$$

The electric field components are obtained using (12-10a) - (12-10c)

Thus $E_r = 0$

$$E_\theta = -j\frac{ke^{-jkr}}{4\pi r}\left[L_\phi + \eta N_\theta\right] = +j\frac{abE_o e^{-jkr}}{2\lambda r}\sin\phi\cdot e^{j\Upsilon}\frac{\sin\Upsilon}{\Upsilon}\cdot\frac{\sin^2\left(\frac{ka}{4}\sin\theta\cos\phi\right)}{\left(\frac{ka}{4}\cdot\sin\theta\cdot\cos\phi\right)^2}$$

$$E_\phi = j\frac{ke^{-jkr}}{4\pi r}\left[L_\theta - \eta N_\phi\right] = j\frac{abE_o e^{-jkr}}{2\lambda r}\cos\theta\cos\phi\, e^{j\Upsilon}\frac{\sin\Upsilon}{\Upsilon}\cdot\frac{\sin^2\left(\frac{ka}{4}\sin\theta\cos\phi\right)}{\left(\frac{ka}{4}\sin\theta\cdot\cos\phi\right)^2}$$

According (12-13a)

$$U(\theta,\phi) = \frac{|E_o|^2}{2\eta}\cdot\left(\frac{ab}{2\lambda}\right)^2\cdot\left\{\left(\sin^2\phi + \cos^2\theta\cos^2\phi\right)\left[\frac{\sin\Upsilon}{\Upsilon}\cdot\frac{\sin^2\left(\frac{ka}{4}\sin\theta\cos\phi\right)}{\left(\frac{ka}{4}\sin\theta\cdot\cos\phi\right)^2}\right]\right\}$$

$$U(\theta,\phi)\Big|_{max} = U_{max}(\theta=0) = \frac{|E_o|^2(ab)^2}{8\eta\lambda^2}$$

Using (12-39a)

$$P_{rad} = \oint\!\!\!\oint_S \underline{W}_{ave}\cdot d\underline{s} = \frac{|E_o|^2}{2\eta}\left\{\int_0^b\int_{-a/2}^{0}\left(\frac{2}{a}x'+1\right)\hat{x}\,dx'dy' + \int_0^b\int_0^{a/2}\left(-\frac{2}{a}x'+1\right)\hat{x}\,dx'dy'\right\}$$

$$P_{rad} = \frac{|E_o|^2}{2\eta}\left\{2b\int_{-a/2}^{0}\left(\frac{2}{a}x'+1\right)^2 dx'\right\} = \frac{|E_o|^2 ab}{6\eta}$$

$$D_o = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi\left[\frac{|E_o|^2}{8\eta}\left(\frac{ab}{\lambda}\right)^2\right]}{\frac{|E_o|^2 ab}{6\eta}} = \frac{3\pi ab}{\lambda^2}$$

Using (12-40)

$$\epsilon_{ap} = \frac{A_{em}}{A_p} = \frac{\frac{\lambda^2}{4\pi}D_o}{ab} = \frac{\lambda^2}{4\pi ab}D_o = \frac{\lambda^2}{4\pi ab}\frac{3\pi ab}{\lambda^2} = \frac{3}{4} = 0.75 = 75\%$$

as compared to 81% for the cosine distribution.

b. $\underline{E}_a = \hat{a}_y E_o\cos^2\left(\frac{\pi}{a}x'\right) = \hat{a}_y E_y$

$E_y = E_o\cos^2\left(\frac{\pi}{a}x'\right)$

$\underline{M}_s = -2\hat{n}\times\underline{E}_a = -2\hat{a}_z\times\hat{a}_y E_y = \hat{a}_x 2E_y$

$J_x = J_y = J_z = M_y = M_z = 0$

$M_x = 2E_y = 2E_o\cos^2\left(\frac{\pi}{a}x'\right)$

$N_\theta = N_\phi = 0$

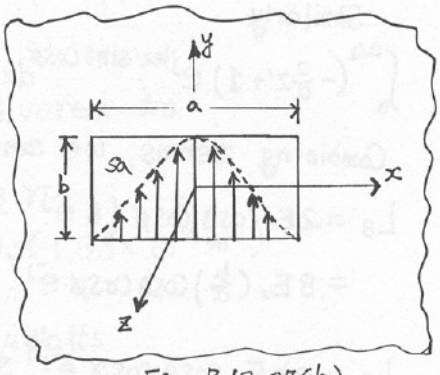


Fig. P 12.27(b)

$$L_\theta = \iint_{S_a} M_x\cos\theta\cos\phi\, e^{jkr'\cos\psi}\,ds'$$

$$L_\theta = 2E_o\cos\theta\cos\phi\int_{-a/2}^{a/2}\cos^2\left(\frac{\pi}{a}x'\right)e^{jkx'\sin\theta\cos\phi}\,dx'\int_{-b/2}^{b/2}e^{jky'\sin\theta\sin\phi}\,dy'$$

$$\int_{-b/2}^{b/2}e^{jky'\sin\theta\sin\phi}\,dy' = b\,\frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\frac{kb}{2}\sin\theta\sin\phi} = b\frac{\sin\Upsilon}{\Upsilon}, \qquad \Upsilon = \frac{kb}{2}\sin\theta\sin\phi$$

(Continued)

12-27 (Cont'd)

Using the identity

$$\int e^{\alpha x}\cos^2(\beta x)\,dx = \frac{2\beta^2 e^{\alpha x}}{\alpha[\alpha^2+(2\beta)^2]}$$

We can write that

$$\int_{-a/2}^{a/2} e^{jkx'\sin\theta\cos\phi}\cos^2\left(\frac{\pi}{a}x'\right)dx' = \frac{a}{2}\frac{\pi^2}{\pi^2-\left(\frac{ka}{2}\sin\theta\cos\phi\right)^2}\frac{\sin\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{\frac{ka}{2}\sin\theta\cos\phi}$$

$$= \frac{a}{2}\frac{\pi^2}{\pi^2-(X)^2}\cdot\frac{\sin(X)}{X},\quad X=\frac{ka}{2}\sin\theta\cos\phi$$

Thus

$$L_\theta = abE_o\cos\theta\cos\phi\,\frac{\pi^2}{\pi^2-(X)^2}\cdot\frac{\sin X}{X}\cdot\frac{\sin Y}{Y},\quad X=\frac{ka}{2}\sin\theta\cos\phi,\ Y=\frac{kb}{2}\sin\theta\sin\phi$$

In a similar manner

$$L_\phi = \iint_{-Sa} -M_x\sin\phi\, e^{ikr'\cos\psi}\,ds'$$

$$L_\phi = -abE_o\sin\phi\,\frac{\pi^2}{\pi^2-(X)^2}\cdot\frac{\sin X}{X}\cdot\frac{\sin Y}{Y}$$

Thus

$$E_r = 0$$

$$E_\theta = -j\frac{ke^{-jkr}}{4\pi r}[L_\phi+\eta\cancel{N_\theta}^{\,0}] = j\frac{abE_o e^{-jkr}}{2\lambda r}\sin\phi\,\frac{\pi^2}{\pi^2-(X)^2}\cdot\frac{\sin X}{X}\cdot\frac{\sin Y}{Y}$$

$$E_\phi = j\frac{ke^{-jkr}}{4\pi r}[L_\theta-\eta\cancel{N_\phi}^{\,0}] = j\frac{abE_o e^{-jkr}}{2\lambda r}\cos\theta\cos\phi\,\frac{\pi^2}{\pi^2-(X)^2}\cdot\frac{\sin X}{X}\cdot\frac{\sin Y}{Y}$$

$$U(\theta,\phi) = \frac{(ab)^2|E_o|^2}{8\eta\lambda^2}[\sin^2\phi+\cos^2\theta\cos^2\phi]\,\frac{\pi^2}{\pi^2-(X)^2}\cdot\frac{\sin X}{X}\cdot\frac{\sin Y}{Y}$$

$$X=\frac{ka}{2}\sin\theta\cos\phi,\quad Y=\frac{kb}{2}\sin\theta\sin\phi$$

$$U_{max}(\theta=0) = \frac{(ab)^2|E_o|^2}{8\eta\lambda^2}$$

Using (12-39a)

$$P_{rad} = \iint_S \underline{W}_{ave}\cdot d\underline{s} = \frac{|E_o|^2}{2\eta}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}\cos^4\left(\frac{\pi}{a}x'\right)dx'dy' = \frac{b|E_o|^2}{2\eta}\int_{-a/2}^{a/2}\cos^4\left(\frac{\pi}{a}x'\right)dx'$$

Since $\int_{-a/2}^{a/2}\cos^4\left(\frac{\pi}{a}x'\right)dx' = \frac{3a}{8}$

$$P_{rad} = \frac{3}{16}\frac{ab|E_o|^2}{\eta},\quad \text{and}$$

$$D_o = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi\left[\frac{(ab)^2|E_o|^2}{8\eta\lambda^2}\right]}{\frac{3}{16}\frac{ab|E_o|^2}{\eta}} = \frac{8\pi}{3}\cdot\frac{ab}{\lambda^2}$$

Thus

$$\epsilon_{ap} = \frac{A_{em}}{A_P} = \frac{\lambda^2}{4\pi}\frac{D_o}{ab} = \frac{\lambda^2}{4\pi ab}D_o = \frac{\lambda^2}{4\pi ab}\frac{8\pi ab}{3\lambda^2} = \frac{2}{3} = 66.67\%$$

as compared to 81 % for the cosine distribution.

(Continued)

12-27 (Cont'd)

c. $\underline{E}_a = \hat{a}_y \cdot E_0 \cdot \cos(\frac{\pi}{a}x')\cos(\frac{\pi}{b}y') = \hat{a}_y E_y$, $E_y = E_0 \cos(\frac{\pi}{a}x')\cos(\frac{\pi}{b}y')$

$N_\theta = N_\phi = 0$

$$L_\theta = 2E_0\cos\theta\cos\phi \int_{-a/2}^{a/2} \cos(\frac{\pi x'}{a}) e^{jkx'\sin\theta\cos\phi} dx' \int_{-b/2}^{b/2} \cos(\frac{\pi y'}{b}) e^{jky'\sin\theta\sin\phi} dy'$$

$$= 2E_0\cos\theta\cdot\cos\phi \left[\int_{-b/2}^{b/2} \cos(\frac{\pi y'}{b}) e^{jky'\sin\theta\sin\phi} dy'\right]$$

$$\times \left[-(\frac{\pi a}{2}) \frac{\cos(\frac{ka}{2}\sin\theta\cdot\cos\phi)}{(\frac{ka}{2}\sin\theta\cos\phi)^2 - (\frac{\pi}{2})^2}\right] = 2E_0\cos\theta\cos\phi\cdot a\cdot b\cdot\frac{\pi^2}{4}\cdot\frac{\cos X}{X^2-(\frac{\pi}{2})^2}\cdot\frac{\cos Y}{Y^2-(\frac{\pi}{2})^2}.$$

$X = \frac{ka}{2}\sin\theta\cos\phi$, $Y = \frac{kb}{2}\sin\theta\sin\phi$.

In similar procedure

$$L_\phi = -2E_0\cdot\sin\phi\cdot a\cdot b\,\frac{\pi^2}{4}\cdot\frac{\cos X}{X^2-(\frac{\pi}{2})^2}\cdot\frac{\cos Y}{Y^2-(\frac{\pi}{2})^2}.$$

The fields are

$$E_\theta = -j\frac{ke^{-jkr}}{4\pi r}\cdot L_\phi = j\frac{k\cdot a\cdot b\,E_0}{2\pi r}\sin\phi\left(\frac{\pi^2}{4}\right)\frac{\cos X}{X^2-(\frac{\pi}{2})^2}\cdot\frac{\cos Y}{Y^2-(\frac{\pi}{2})^2}$$

$$E_\phi = j\frac{ke^{jkr}}{4\pi r}\cdot L_\theta = j\frac{ka\cdot bE_0}{2\pi r}\cos\theta\cos\phi\cdot\left(\frac{\pi^2}{4}\right)\cdot\frac{\cos X}{X^2-(\frac{\pi}{2})^2}\cdot\frac{\cos Y}{Y^2-(\frac{\pi}{2})^2}$$

$U_{max}$ occurs at $\theta = 0°$.

$$U_{max} = r^2\frac{1}{2}\frac{|E|^2}{\eta} = \frac{1}{2}\cdot\frac{1}{\eta}\cdot\left(\frac{k^2a^2b^2}{4\pi^2}E_0^2\right)\cdot\left(\frac{\pi^2}{4}\right)^2\left(\frac{2}{\pi}\right)^4\left(\frac{2}{\pi}\right)^4 = \frac{|E_0|^2}{\eta\lambda^2}\frac{8}{\pi^4}a^2b^2$$

Using (12-39a).

$$P_{rad} = \oint\oint_{Wave}\cdot ds = \frac{|E_0|^2}{2\eta}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}\cos^2\frac{\pi}{a}x'\cdot\cos^2\frac{\pi}{b}y'\,dx'dy' = \frac{|E_0|^2}{2\eta}\cdot\frac{b}{2}\cdot\frac{a}{2}$$

$$P_{rad} = \frac{|E_0|^2}{8\eta}\cdot a\cdot b, \quad \text{Thus} \quad D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi\frac{|E_0|^2}{\eta\lambda^2}\cdot\frac{8}{\pi^4}\cdot a^2b^2}{\frac{|E_0|^2}{8\eta}\cdot a\cdot b} = \frac{64}{\pi^4}\frac{ab}{\lambda^2}\cdot 4\pi$$

$$\epsilon_{ap} = \frac{A_{em}}{A_p} = \frac{\frac{\lambda^2}{4\pi}D_0}{ab} = \frac{\lambda^2}{4\pi ab}D_0 = \frac{\lambda^2}{4\pi ab}\frac{64}{\pi^4}\cdot\frac{ab}{\lambda^2}\cdot 4\pi = \frac{64}{\pi^4}$$

$$\epsilon_{ap} = \frac{64}{\pi^4} = 0.657 = 65.7\%$$

d. $\underline{E}_a = \hat{a}_y E_0\cos^2(\frac{\pi}{a}x')\cos^2(\frac{\pi}{b}y') = \hat{a}_y E_y$, $E_y = E_0\cos^2(\frac{\pi}{a}x')\cos^2(\frac{\pi}{b}y')$

Using the coordinate system of Fig. 12-15(b) and the same procedure as in part (b), we have that, $N_\theta = N_\phi = 0$

$$L_\theta = 2E_0\cos\theta\cos\phi\int_{-a/2}^{a/2}\cos^2(\frac{\pi}{a}x')e^{jkx'\sin\theta\cos\phi}dx'\int_{-b/2}^{b/2}\cos^2(\frac{\pi}{b}y')e^{jky'\sin\theta\sin\phi}dy'$$

(Continued)

12-27 (Cont'd) $L_\theta = \frac{ab}{2} E_0 \cos\theta \cos\phi \frac{\pi^2}{\pi^2-(X)^2} \frac{\pi^2}{\pi^2-(Y)^2} \cdot \frac{\sin X}{X} \cdot \frac{\sin Y}{Y}$

$L_\phi = \frac{ab}{2} E_0 \sin\phi \frac{\pi^2}{\pi^2-(X)^2} \cdot \frac{\pi^2}{\pi^2-(Y)^2} \cdot \frac{\sin X}{X} \cdot \frac{\sin Y}{Y}$, $X = \frac{ka}{2} \sin\theta\cos\phi$

Thus $Y = \frac{kb}{2} \sin\theta \cdot \sin\phi$

$E_\theta = j \frac{ab E_0 e^{-jkr}}{4\lambda r} \sin\phi \frac{\pi^2}{\pi^2-(X)^2} \frac{\pi^2}{\pi^2-(Y)^2} \cdot \frac{\sin X}{X} \cdot \frac{\sin Y}{Y}$.

$E_\phi = j \frac{ab E_0 e^{-jkr}}{4\lambda r} \cos\theta \cdot \cos\phi \frac{\pi^2}{\pi^2-(X)^2} \frac{\pi^2}{\pi^2-(Y)^2} \cdot \frac{\sin X}{X} \cdot \frac{\sin Y}{Y}$

$U(\theta,\phi) = \frac{(ab)^2 |E_0|^2}{32\eta\lambda^2} [\sin^2\phi + \cos^2\theta \cos^2\phi] \frac{\pi^2}{\pi^2-(X)^2} \cdot \frac{\pi^2}{\pi^2-(Y)^2} \cdot \frac{\sin X}{X} \frac{\sin Y}{Y}$

$U_{max}$ occurs at $\theta = 0°$. $U_{max}(\theta=0°) = (ab)^2 |E_0|^2 / (32\eta\lambda^2)$

$P_{rad} = \frac{|E_0|^2}{2\eta} \int_{-a/2}^{a/2} \cos^4(\frac{\pi}{a}x')dx' \int_{-b/2}^{b/2} \cos^4(\frac{\pi}{b}y')dy'$, $\left(\leftarrow \int_{-a/2}^{a/2} \cos^4(\frac{\pi}{a}x')dx' = \frac{3a}{8}\right)$

$P_{rad} = \frac{9}{128} \frac{ab|E_0|^2}{\eta}$. Thus $D_0 = \frac{4\pi U_{max}}{P_{rad}} = 4\pi \left[\frac{(ab)^2|E_0|^2}{32\eta\lambda^2}\right] \Big/ \left[\frac{9}{128} \cdot \frac{ab|E_0|^2}{\eta}\right] = \frac{16\pi ab}{9\lambda^2}$

$\epsilon_{ap} = \frac{A_{em}}{A_P} = \frac{\frac{\lambda^2}{4\pi} D_0}{ab} = \frac{\lambda^2}{4\pi ab} D_0 = \frac{\lambda^2}{4\pi ab} \frac{16\pi ab}{9\lambda^2} = \frac{4}{9} = 44.44\%$

12-28. $a = 0.9'' = 2.286\text{ cm} = 0.763\lambda$

$b = 0.4'' = 1.02\text{ cm} = 0.340\lambda$

a. Using the computer program at the end of Chapter 2.

$D_0 = 4.264 = 6.298\text{ dB}$

b. From Table 12.1

$D_0 = 0.81 [4\pi (0.763)(0.340)] = 2.64 = 4.217\text{ dB}$

12-29. $a = 0.42'' = 1.067\text{ cm} = 0.711\lambda$

$b = 0.17'' = 0.432\text{ cm} = 0.288\lambda$

a. Using the computer program at the end of chapter 2.

$D_0 = 3.981 = 5.999\text{ dB}$

b. From Table 12.1

$D_0 = 0.81 [4\pi (0.711)(0.288)] = 2.084 = 3.189\text{ dB}$

**12-30.** $a = 0.9'' = 2.286\ cm = 0.7620\ \lambda$

$b = 0.4'' = 1.016\ cm = 0.3387\ \lambda$

$d = 0.85\lambda$

$R_o = -30\ dB = 31.6228$

$f = 10\ GHz \Rightarrow \lambda = 30\times10^9/(10\times10^9) = 3\ cm$

$D_t = D_e\, D_a$

$$D_e = 0.81\left[4\pi\frac{ab}{\lambda^2}\right] = 0.81(4\pi)\frac{(0.7620\lambda)(0.3387\lambda)}{\lambda^2} = 2.6270 = 4.1947\ dB$$

$$D_a = \frac{2R_o^2}{1+(R_o^2-1)f\frac{\lambda}{L+d}}$$

$$f = 1+0.636\left\{\frac{2}{R_o}\cosh\left[\sqrt{(\cosh^{-1}R_o)^2-\pi^2}\ \right]\right\}^2$$

$$= 1+0.636\left\{\frac{2}{31.6228}\cosh\left[\sqrt{(4.1468)^2-\pi^2}\ \right]\right\}^2 = 1.1440$$

$$\cosh^{-1}R_o = \cosh^{-1}(31.6228) = \ln\left[31.6228\pm\sqrt{(31.6228)^2-1}\ \right]$$
$$= \ln(63.2297) = 4.1468$$

$$L = 3(0.85\lambda) = 2.55\lambda$$

$$D_a = \frac{2(31.6228)^2}{1+[(31.6228)^2-1](1.144)\frac{\lambda}{(2.55+0.85)\lambda}} = \frac{2(1{,}000)}{1+336.1346}$$

$$= 5.9323 = 7.7322\ dB$$

$$D_t = D_e D_a = 2.6270\cdot(5.9323) = 15.5843 = 11.9269\ dB$$

**12-31.** $D_t = D_w[\text{Waveguide}]\, D_a[\text{array}]$

From Table 12.1.

$$D_w = \frac{8}{\pi^2}\left[4\pi\frac{ab}{\lambda^2}\right] = 0.81\left[4\pi\frac{ab}{\lambda^2}\right]$$

$$\lambda = \frac{30\times10^9}{10\times10^9} = 3\ cm = 1.1811'' \Rightarrow a = \frac{0.9}{1.1811}\lambda = 0.7620\lambda,\quad b = \frac{0.4}{1.1811}\lambda = 0.3387\lambda$$

$$D_w = 0.81\left[4\pi\frac{0.7620\lambda(0.3387\lambda)}{\lambda^2}\right] = 0.81(3.2429) = 2.6268 = 4.1942\ dB$$

$$D_a = \pi\cos\theta_o\, D_x D_y\Big|_{\theta_o=0^\circ} = \pi D_x D_y \qquad (\leftarrow \text{From eq (6-100)})$$

$$D_x = D_y \simeq 2N\left(\frac{d}{\lambda}\right) = 2(8)\left(\frac{0.85\lambda}{\lambda}\right) = 13.6 \qquad (\leftarrow \text{From Table 6-7})$$

$$D_a = \pi D_x D_y = \pi(13.6)(13.6) = 581.069 = 581.07 = 27.642\ dB$$

$$D_t \simeq D_w D_a = 2.6268(581.069) = 1{,}526.35 = 31.84\ dB$$

$$= 4.1942(dB) + 27.642\ (dB) = 31.836\ dB$$

12-32. The results can be obtained by using a comparative analogy between the fields of a rectangular aperture when not mounted and mounted on an infinite ground plane. These are listed in Table 12.1.

Therefore we can write that the fields of the circular aperature of Section 12.6.1 , when it is not mounted on an infinite ground plane, are [by using (12-53a) - (12-53c)]

$$E_r \simeq 0$$

$$E_\theta \simeq j\frac{ka^2E_0e^{jkr}}{2r}\left\{(1+\cos\theta)\sin\phi\cdot\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right\}$$

$$E_\phi \simeq j\frac{ka^2E_0e^{jkr}}{2r}\left\{(1+\cos\theta)\cos\phi\cdot\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right\}$$

$$H_r \simeq 0$$

$$H_\theta = -E_\phi/\eta$$

$$H_\phi = E_\theta/\eta$$

12-33.

$$E_\rho = \frac{E_0}{\rho'}J_1\left(\chi'_{11}\frac{\rho'}{a}\right)\sin\phi'$$

$$E_\phi = E_0J_1'\left(\chi'_{11}\frac{\rho'}{a}\right)\cos\phi', \qquad J_1' = \frac{\partial}{\partial\rho'}J_1$$

$$E_z = 0$$

a. Using (VII-7b)

$$E_x = E_\rho\cos\phi' - E_\phi\sin\phi' = \frac{E_0}{2}\left\{\frac{1}{\rho'}J_1\left(\chi'_{11}\frac{\rho'}{a}\right)\sin 2\phi' - J_1'\left(\chi'_{11}\frac{\rho'}{a}\right)\sin 2\phi'\right\}$$

$$E_y = E_\rho\sin\phi' + E_\phi\cos\phi' = E_0\left\{\frac{1}{\rho'}J_1\left(\chi'_{11}\frac{\rho'}{a}\right)\sin^2\phi' + J_1'\left(\chi'_{11}\frac{\rho'}{a}\right)\cos^2\phi'\right\}$$

$$L_y = \iint_{S_a} M_y e^{jkr'\cos\psi}ds' = \iint_{S_a} M_y e^{jk\rho'\cos\psi}\rho'd\rho'd\phi'$$

$$L_x = \iint_{S_a} M_x e^{jkr'\cos\psi}ds' = \iint_{S_a} M_x e^{jk\rho'\cos\psi}\rho'd\rho'd\phi'$$

$$\underline{M} = -2\hat{n}\times E_a = -2\hat{a}_z\times(\hat{a}_xE_x + \hat{a}_yE_y) = -\hat{a}_y2E_x + \hat{a}_x2E_y$$

$$M_x = 2E_y\,, \quad M_y = -2E_x$$

$$\underline{J} = 0 \Rightarrow J_x = J_y = J_z = 0$$

Therefore

$$L_y = -E_0\int_0^{2\pi}\int_0^a\left\{\frac{1}{\rho'}J_1\left(\chi'_{11}\frac{\rho'}{a}\right)\sin 2\phi' - J_1'\left(\chi'_{11}\frac{\rho'}{a}\right)\sin 2\phi'\right\}e^{jk\rho'\cos\psi}\rho'd\rho'd\phi'$$

$$= -E_0\int_0^a J_1\left(\chi'_{11}\frac{\rho'}{a}\right)\left\{\int_0^{2\pi}\sin 2\phi'\cdot e^{jk\rho'\cos\psi}d\phi'\right\} + E_0\int_0^a J_1'\left(\chi'_{11}\frac{\rho'}{a}\right)\int_0^{2\pi}\sin 2\phi'\cdot e^{jk\rho'\cos\psi}\rho'd\phi'd\rho'$$

(Continued)

12-33 (Cont'd) However

$$I_o=\int_0^{2\pi}\sin 2\phi\, e^{jk\rho'\cos\psi}d\phi'=\frac{1}{2j}\left\{\int_0^{2\pi}e^{j(k\rho'\sin\theta\cos(\phi'-\phi)+2\phi')}d\phi'\right.$$

$$\left.-\int_0^{2\pi}e^{j(k\rho'\sin\theta\cos(\phi'-\phi)-2\phi')}d\phi'\right\}$$

Letting $\phi'-\phi=\beta \Rightarrow d\phi'=d\beta$, we can write

$$I_o=\frac{1}{2j}\left\{e^{j2\phi}\int_{-\phi}^{2\pi-\phi}e^{j(k\rho'\sin\theta\cos\beta+2\beta)}d\beta-e^{-j2\phi}\int_{-\phi}^{2\pi-\phi}e^{j(k\rho'\sin\theta\cos\beta-2\beta)}d\beta\right\}$$

with the aid of (V-35), it reduces to

$$I_o=\frac{1}{2j}\left\{e^{j2\phi}\cdot 2\pi\cdot J_2(k\rho'\sin\theta)+e^{-j2\phi}\cdot 2\pi\cdot J_{-2}(k\rho'\sin\theta)\right\}$$

Using (V-10) we can write it as

$I_o=\int_0^{2\pi}\sin 2\phi\, e^{jk\rho'\cos\psi}d\phi'=-2\pi\cdot\sin 2\phi\cdot J_2(k\rho'\sin\theta)$, Thus

$$L_y=E_o\cdot 2\pi\cdot\sin 2\phi\int_0^a\rho' J_2(k\rho'\sin\theta)\left[\frac{1}{\rho'}J_1(\chi_{11}'\frac{\rho'}{a})-J_1'(\chi_{11}'\frac{\rho'}{a})\right]d\rho'$$

Using (V-19), it reduces to

$$L_y=\frac{2\pi\chi_{11}'\sin 2\phi}{a}E_o\int_0^a\rho' J_2(k\rho'\sin\theta)J_2(\chi_{11}'\frac{\rho'}{a})d\rho'$$

Since $\int xJ_p(\alpha x)J_p(\gamma x)dx=\dfrac{\gamma xJ_p(\alpha x)J_{p-1}(\gamma x)-\alpha xJ_{p-1}(\alpha x)J_p(\gamma x)}{\alpha^2-\gamma^2}$

then

$$L_y=E_o\cdot\frac{2\pi\cdot\chi_{11}'\cdot\sin 2\phi}{a}\cdot\left[\frac{\chi_{11}'J_2(ka\sin\theta)J_1(\chi_{11}')-ka\sin\theta J_1(ka\sin\theta)J_2(\chi_{11}')}{(k\sin\theta)^2-(\chi_{11}'/a)^2}\right\}$$

Because $J_1'(\chi_{11}')=-J_2(\chi_{11}')-\frac{1}{\chi_{11}'}J_1(\chi_{11}')=0\Rightarrow J_1(\chi_{11}')=\chi_{11}'J_2(\chi_{11}')$

$$L_y=-2\pi a\cdot E_o\cdot\sin 2\phi\cdot\frac{J_1(\chi_{11}')}{\chi_{11}'^2}\left\{\frac{(\chi_{11}')^2J_2(ka\sin\theta)-ka\sin\theta J_1(ka\sin\theta)}{1-(ka\sin\theta/\chi_{11}')^2}\right\}$$

Using a similar procedure, it can be shown that

$$L_x=\iint_{Sa}M_x e^{jk\rho'\cos\psi}\rho'd\rho'd\phi'=2E_o\int_0^{2\pi}\int_0^a\left\{\frac{1}{\rho'}J_1(\chi_{11}'\frac{\rho'}{a})\sin^2\phi'+J_1'(\chi_{11}'\frac{\rho'}{a})\cos^2\phi'\right\}e^{jk\rho'\cos\psi}\rho'd\rho'd\phi'$$

$$=\frac{-4\pi a\,J_1(\chi_{11}')E_o}{(\chi_{11}')^2}\left\{\frac{\sin^2\phi\,J_1(ka\sin\theta)[ka\sin\theta-\frac{(\chi_{11}')^2}{ka\sin\theta}]-\cos^2\phi\,(\chi_{11}')^2J_1'(ka\sin\theta)}{[1-(ka\sin\theta/\chi_{11}')^2]}\right\}$$

$L_\theta=L_x\cos\theta\cos\phi+L_y\cos\theta\sin\phi$, $L_\phi=-L_x\sin\phi+L_y\cos\phi$

$$L_\theta=4\pi a\,J_1(\chi_{11}')E_o\cos\theta\cos\phi\;\frac{J_1'(ka\sin\theta)}{[1-(ka\sin\theta/\chi_{11}')^2]},\quad L_\phi=-4\pi a J_1(\chi_{11}')E_o\sin\phi\;\frac{J_1(ka\sin\theta)}{ka\sin\theta}$$

Thus using (12-10a)-(12-10f)

$$E_r\simeq 0$$

$$E_\theta\simeq j\cdot\frac{kaE_oJ_1(\chi_{11}')e^{-jkr}}{r}\sin\phi\cdot\frac{J_1(ka\sin\theta)}{ka\sin\theta}$$

$$E_\phi\simeq j\frac{kaE_oJ_1(\chi_{11}')e^{-jkr}}{r}\cdot\cos\theta\cos\phi\cdot\frac{J_1'(ka\sin\theta)}{[1-(ka\sin\theta/\chi_{11}')^2]}$$

$$H_r\simeq 0$$

$$H_\theta\simeq -E_\phi/\eta$$

$$H_\phi\simeq E_\theta/\eta$$

(Continued)

12-33 (Cont'd)

b. The results can be obtained by using a comparative analogy between the fields radiated by a rectangular aperture when not mounted and mounted on an infinite ground plane. These are listed in Table 12.1. Also compare Problem 12.32.

Therefore we can write that the fields of the circular aperture of Section 12.6.2, when it is not mounted on an infinite ground plane, are (by using the results of part a)

$$E_r \simeq 0$$

$$E_\theta \simeq j\frac{ka E_0 J_1(x_{11}) e^{-jkr}}{2r}(1+\cos\theta)\sin\phi \frac{J_1(ka\sin\theta)}{ka\sin\theta} \qquad H_r \simeq 0$$

$$E_\phi \simeq j\frac{ka E_0 J_1(x_{11}) e^{-jkr}}{2r}(1+\cos\theta)\cos\phi \frac{J_1'(ka\sin\theta)}{[1-(ka\sin\theta/x_{11})^2]}, \qquad H_\phi \simeq E_\theta/\eta$$

12-34. $E_\theta = \frac{C_2}{2}\sin\phi(1+\cos\theta)\cdot\frac{J_1(z)}{z}$

$$E_\phi = \frac{C_2}{2}\cos\phi(1+\cos\theta)\frac{J_1'(z)}{1-(z/x_{11})^2}$$

$z = ka\sin\theta$, $x_{11} = 1.841$, $J_1'(z) = J_0(z) - J_1(z)/z$

$$\underline{E}_t = \left[\hat{a}_\theta \cdot \frac{\sin\phi}{2}(1+\cos\theta) + \hat{a}_\phi \frac{\cos\phi}{2}(1+\cos\theta)\right]$$

$$= \frac{1+\cos\theta}{2}\left[\hat{a}_\theta \sin\phi + \hat{a}_\phi \cos\phi\right]$$

$$E_r = (\hat{a}_\theta \sin\phi + \hat{a}_\phi \cos\phi \sin\phi)$$

$$\hat{a}_t = \frac{(\hat{a}_\theta \sin\phi + \hat{a}_\phi \cos\phi)}{\sqrt{\sin^2\phi + \cos^2\phi}} = (\hat{a}_\theta \sin\phi + \hat{a}_\phi \cos\phi)$$

$$\hat{a}_r = \frac{\hat{a}_\theta \sin\phi + \hat{a}_\phi \cos\theta \cos\phi}{\sqrt{\sin^2\phi + \cos^2\theta \cos^2\phi}}$$

$$PLF = |\hat{a}_t \cdot \hat{a}_r|^2 = \left|\frac{\sin^2\phi + \cos\theta \cos^2\phi}{\sqrt{\sin^2\phi + \cos^2\theta \cos^2\phi}}\right|^2 = \frac{(\sin^2\phi + \cos\theta \cos^2\phi)^2}{\sin^2\phi + \cos^2\theta \cos^2\phi}$$

12-35. Using the computer program at the end of Chapter 2.

a. $a = 0.5\lambda \Rightarrow D_o = 11.24 = 10.51\,dB$

b. $a = 1.5\lambda \Rightarrow D_o = 92.704 = 19.67\,dB$

c. $a = 3.0\lambda \Rightarrow D_o = 357.278 = 25.53\,dB$

Using Table 12.2

a. $a = 0.5\lambda \Rightarrow D_o = (2\pi a/\lambda)^2 = [2\pi(0.5)]^2 = 9.8696 = 9.943\,dB$

b. $a = 1.5\lambda \Rightarrow D_o = (2\pi a/\lambda)^2 = [2\pi(1.5)]^2 = 88.826 = 19.485\,dB$

c. $a = 3.0\lambda \Rightarrow D_o = (2\pi a/\lambda)^2 = [2\pi(3)]^2 = 355.306 = 25.506\,dB$

They compare quite well except for $a = 0.5\lambda$

12-36 Using the computer program at the end of Chapter 2.

a. $a = 0.5\lambda \Rightarrow D_o = 11.264 = 10.517\,dB$

b. $a = 1.5\lambda \Rightarrow D_o = 93.197 = 19.694\,dB$

c. $a = 3.0\lambda \Rightarrow D_o = 356.815 = 25.524\,dB$

12-37. With ground plane

Using the computer program at the end of Chapter 2.

a. $a = 0.5\lambda \Rightarrow D_o = 9.922 = 9.966\,dB$

b. $a = 1.5\lambda \Rightarrow D_o = 75.161 = 18.759\,dB$

c. $a = 3.0\lambda \Rightarrow D_o = 295.096 = 24.699\,dB$

Using Table 12.2

a. $a = 0.5\lambda \Rightarrow D_o = 0.836(2\pi a/\lambda) = 0.836(9.8696) = 8.251 = 9.165\,dB$

b. $a = 1.5\lambda \Rightarrow D_o = 0.836(2\pi a/\lambda) = 0.836(88.826) = 74.259 = 18.707\,dB$

c. $a = 3.0\lambda \Rightarrow D_o = 0.836(2\pi a/\lambda) = 0.836(355.306) = 297.036 = 24.728\,dB$

Without ground plane

Using the computer program at the end of Chapter 2.

a. $a = 0.5\lambda \Rightarrow D_o = 8.8244 = 9.457\,dB$

b. $a = 1.5\lambda \Rightarrow D_o = 75.9458 = 18.805\,dB$

c. $a = 3.0\lambda \Rightarrow D_o = 296.959 = 24.727\,dB$

12-38. From Table 12.2 From Figs. 12.19 & 12.20

a. HPBW (E-plane) = 29.2/1.5 = 19.47° HPBW ≃ 20°

b. HPBW (H-plane) = 37.0/1.5 = 24.67° HPBW ≃ 23.8°

c. FNBW (E-plane) = 69.9/1.5 = 46.6° FNBW ≃ 49°

d. FNBW (H-plane) = 98.0/1.5 = 65.33° FNBW ≃ 68°

e. FSLMM (E-plane) = −17.6 dB FSLMM ≃ −19 dB

f. FSLMM (H-plane) = −26.2 dB FSLMM ≃ −28.5 dB

12-39. a. $\underline{E}_a = \hat{a}_y E_o [1-(\rho'/a)^2] \Rightarrow \underline{M}_s = -2\hat{n} \times \underline{E}_a = \hat{a}_x 2E_o [1-(\rho'/a)^2]$

$$L_\theta = \iint_S (M_x \cos\theta\cos\phi + \cancel{M_y}^{\,0}\cos\theta\sin\phi - \cancel{M_z}^{\,0}\sin\theta)\, e^{jkr'\cos\psi} ds'$$

$$= 2E_o \cos\theta\cos\phi \int_0^a \rho'[1-(\rho'/a)^2] \cdot \int_0^{2\pi} e^{jk\rho'\sin\theta'\cos\phi'} d\phi' d\rho'$$

Using (V-35) it reduces, after separating into two integrals to

$$L_\theta = 4\pi E_o \cos\theta\cos\phi \left[\int_0^a \rho' J_0(k\rho'\sin\theta) d\rho' - \frac{1}{a^2}\int_0^a \rho'^3 J_0(k\rho'\sin\theta) d\rho'\right]$$

Making a change of variable of the form

$$x = k\rho'\sin\theta \Rightarrow dx = k\cdot\sin\theta\, d\rho'$$

We can write it as

$$L_\theta = 4\pi E_o \cos\theta\cos\phi \left[\int_0^a \rho' J_0(k\rho'\sin\theta) d\rho' - \frac{1}{a^2}\cdot\frac{1}{(k\sin\theta)^4}\int_0^{ka\sin\theta} x^3 J_0(x) dx\right]$$

Using (V-22) and (V-24) it reduces to

$$L_\theta = 8\pi a^2 E_o \left[\cos\phi\cos\theta \frac{J_2(ka\sin\theta)}{(ka\sin\theta)^2}\right]$$

In a similar manner, it can be shown that

$$L_\phi = \iint_S [-M_x\sin\phi + M_y\cos\phi]\, e^{jkr'\cos\psi} ds' = -8\pi a^2 E_o \left[\sin\phi \frac{J_2(ka\sin\theta)}{(ka\sin\theta)^2}\right]$$

Since $N_\theta = N_\phi = 0$, we can write the radiated fields, using (12-10a)-(12-10f), as

$$E_r \simeq 0$$

$$E_\theta \simeq j2ka^2 E_o \frac{e^{-jkr}}{r}\cdot\left[\sin\phi \frac{J_2(ka\sin\theta)}{(ka\sin\theta)^2}\right]$$

$$E_\phi \simeq j2ka^2 E_o \cdot \frac{e^{-jkr}}{r}\left[\cos\theta\cos\phi \frac{J_2(ka\sin\theta)}{(ka\sin\theta)^2}\right]$$

$$H_r \simeq 0$$

$$H_\theta \simeq -E_\phi/\eta$$

$$H_\phi \simeq E_\theta/\eta$$

(Continued)

12-39. (Cont'd)

b. Following a similar procedure, it can be shown that when

$$E_a = \hat{a}_y E_0 [1-(\rho'/a)^2]^2$$

the radiated field are given by

$$E_r \simeq 0$$

$$E_\theta \simeq j16ka^2E_0 \frac{e^{-jkr}}{r}\left[\sin\phi \frac{J_3(ka\sin\theta)}{(ka\sin\theta)^3}\right]$$

$$E_\phi \simeq j16ka^2E_0 \frac{e^{-jkr}}{r}\left[\cos\theta\cos\phi \frac{J_3(ka\sin\theta)}{(ka\sin\theta)^3}\right]$$

$$H_r \simeq 0,\quad H_\theta \simeq -E_\phi/\eta,\quad H_\phi \simeq E_\theta/\eta$$

12-40 $\underline{E}_a = \hat{a}_y E_0(1-\rho'/a) \Rightarrow \underline{M}_s = \hat{a}_x 2E_0(1-\rho'/a)$, $\underline{J}=0 \rightarrow N_\theta, N_\phi = 0$

$$L_\theta = \iint M_x \cos\theta\cos\phi\, e^{jkr'\cos\psi}\, ds' = 2E_0\cos\theta\cos\phi \int_0^a \rho'(1-\rho'/a)\int_0^{2\pi} e^{jk\rho'\sin\theta\cos(\phi-\phi')}\, d\phi'\, d\rho'$$

Using eq (12-48) $\int_0^{2\pi} e^{jk\rho'\sin\theta\cos(\phi-\phi')}\, d\phi' = 2\pi J_0(k\rho'\sin\theta)$

$$L_\theta = 4\pi E_0\cos\theta\cos\phi\left[\int_0^a \rho' J_0(k\rho'\sin\theta)\, d\rho' - \frac{1}{a}\int_0^a \rho'^2 J_0(k\rho'\sin\theta)\, d\rho'\right]$$

$$L_\theta = 4\pi E_0\cos\theta\cos\phi\left[\frac{\rho'}{k\sin\theta} J_1(k\rho'\sin\theta)\Big|_0^a - \frac{1}{a}\int_0^a \rho'^2 J_0(k\rho'\sin\theta)\, d\rho'\right]$$

Let $X = k\rho'\sin\theta \rightarrow \rho' = \frac{X}{k\sin\theta} \rightarrow d\rho' = \frac{1}{k\sin\theta}\, dx$

$$L_\theta = 4\pi E_0\cos\theta\cos\phi\left\{\frac{a}{k\sin\theta} J_1(ka\sin\theta) - \frac{1}{ak^3\sin^3\theta}\int_0^{ka\sin\theta} X^2 J_0(X)\, dX\right\}$$

Using V-30, V-27

$$L_\theta = 4\pi E_0\cos\theta\cos\phi\left\{\frac{a}{k\sin\theta} J_1(ka\sin\theta) - \frac{X^2 J_1(X)}{ak^3\sin^3\theta}\Big|_0^{ka\sin\theta} + \frac{1}{ak^3\sin^2\theta}\int_0^{ka\sin\theta} X J_1(X)\, dX\right\}$$

$$= 4\pi E_0\cos\theta\cos\phi\left\{\frac{a}{k\sin\theta} J_1(ka\sin\theta) - \frac{a}{k\sin\theta} J_1(ka\sin\theta) + \left[\frac{1}{k^3 a\sin^3\theta}\left\{-XJ_0(X)\Big|_0^{ka\sin\theta} + \int_0^{ka\sin\theta} J_0(X)\, dX\right\}\right]\right\}$$

$$L_\theta = 4\pi E_0\cos\theta\cdot\cos\phi\left\{\frac{1}{k^2\sin^2\theta} J_0(ka\sin\theta) + \int_0^{ka\sin\theta} J_0(X)\, dX\right\}$$

$\int_0^{ka\sin\theta} J_0(X)\, dX$, → This cannot be evaluated in closed form.

$L_\phi$ also has same except for front term.

$$L_\phi = -4\pi E_0\sin\phi\left\{\frac{1}{k^2\sin^2\theta} J_0(ka\sin\theta) + \int_0^{ka\sin\theta} J_0(X)\, dX\right\}$$

12-41 $\underline{E}_a = -\hat{a}_\rho \frac{V}{\epsilon \ln(b/a)} \frac{1}{\rho'} = -\hat{a}_\rho C \frac{1}{\rho'}$ , where $C = \frac{V}{\epsilon \ln(b/a)}$

$\underline{M}_s = -2\hat{n} \times \underline{E}_a = -2\hat{a}_z \times (-\hat{a}_\rho \frac{C}{\rho'}) = \hat{a}_\phi \cdot 2 \cdot \frac{C}{\rho'} \Rightarrow M_\phi = \frac{2C}{\rho'}$

Thus using (12-42c)

$$L_\theta = \iint_S [\cancel{M_\rho}^{0} \cos\theta \cos(\phi-\phi') + M_\phi \cos\theta \sin(\phi-\phi') - \cancel{M_z}^{0} \sin\theta] e^{jkr'\cos\psi} ds'$$

$$= 2C \cos\theta \int_a^b \left[ \int_0^{2\pi} \sin(\phi-\phi') e^{jk\rho'\sin\theta\cos(\phi-\phi')} d\phi' \right] d\rho'$$

Because of azimuthal symmetry, the field is not a function of $\phi$.

choosing $\phi = 0$

$$L_\theta = 2C \cdot \cos\theta \int_a^b \left[ -\int_0^{2\pi} \sin\phi' e^{jk\rho'\sin\theta\cos\phi'} d\phi' \right] d\rho' = 2C \cdot \cos\theta \int_a^b [0]\, d\rho' = 0$$

$$L_\phi = \iint_S [-\cancel{M_\rho}^{0} \sin(\phi-\phi') + M_\phi \cos(\phi-\phi')] e^{jkr'\cos\psi} ds'$$

$$= 2C \int_a^b \left[ \int_0^{2\pi} \cos\phi' e^{jk\rho'\sin\theta\cos\phi'} d\phi' \right] d\rho'$$

$$= 2C \int_a^b \left[ \int_0^{\pi} \cos\phi' e^{jk\rho'\sin\theta\cos\phi'} d\phi' + \int_\pi^{2\pi} \cos\phi' e^{jk\rho'\sin\theta\cos\phi'} d\phi' \right] d\rho'$$

<let $\phi'' = \phi' - \pi \Rightarrow d\phi'' = d\phi'$>

$$L_\phi = 2C \int_a^b \left[ \int_0^{\pi} \cos\phi' e^{jk\rho'\sin\theta\cos\phi'} d\phi' - \int_0^{\pi} \cos\phi'' e^{-jk\rho'\sin\theta\cos\phi''} d\phi'' \right] d\rho'$$

$$= 2C \int_a^b [J_1(k\rho'\sin\theta) - J_1(-k\rho'\sin\theta)]\, d\rho' = 4C \int_a^b J_1(k\rho'\sin\theta)\, d\rho'$$

by using (V-36) and (V-10). With the aid of (V-23)

$$L_\phi = -4C \left[ \frac{1}{k\sin\theta} (J_0(kb\sin\theta) - J_0(ka\sin\theta)) \right]$$

If however the slot is very thin, $L_\phi$ can be approximated by

$$L_\phi = 4C \int_a^b J_1(k\rho'\sin\theta) d\rho' \simeq 4C\, J_1(ka'\sin\theta) \int_a^b d\rho' = 4C(b-a) J_1(ka'\sin\theta)$$

where $a' = (\frac{a+b}{2})$

Using (12-10a) - (12-10f), we can write the radiated fields as

$E_r = 0$

$$E_\theta = -j\frac{k e^{-jkr}}{4\pi r}[L_\phi + \eta \cancel{N_\theta}^{0}] = +j\frac{C e^{-jkr}}{\pi r}\left[\frac{J_0(kb\sin\theta) - J_0(ka\sin\theta)}{\sin\theta}\right]$$

$$\simeq +j\frac{(a-b)Ck e^{-jkr}}{\pi r} J_1(ka'\sin\theta)$$

$$E_\phi = +j\frac{k e^{-jkr}}{4\pi r}[L_\theta - \eta N_\phi] = 0$$

$H_r = 0$, $H_\theta = -E_\phi/\eta$ , $H_\phi = E_\theta/\eta$

12-42 Using Figure 12.21

$$u = ka\sin\theta_1 = \frac{2\pi}{\lambda} a\sin\theta_1 = 3.3 \Rightarrow a = \frac{3.3\lambda}{2\pi\sin(30°)} = 1.05\lambda$$

12-43 $u = ka\sin\theta_1 = \frac{2\pi}{\lambda}(2\lambda)\sin(20°) = 4.298 \simeq 4.3$

Using Figure 12.21 $\Rightarrow$ Beam efficiency $\simeq 97 \sim 98\%$

12-44 a. For a square aperture $\theta_{ce} = \theta_{ch}$. Therefore the optimum dimension

$$a = b = \frac{\lambda}{2\sin(60°)} = \frac{\lambda}{2}\cdot\frac{2}{\sqrt{3}} = \frac{\lambda}{\sqrt{3}} = 0.577\lambda$$

b. The maximum directivity, according to (12-59), is

$$D_0 = \frac{\pi}{\sin^2\theta_c} = \frac{\pi}{\sin^2(60°)} = \frac{4}{3}\pi = 4.189 = 6.221\,\text{dB}$$

c, The directivity at $\theta = 60°$ is $-3.922$ dB from the maximum at $\theta = 0°$

12-45 a. $$a = \frac{\lambda}{3.413\cdot\sin\theta_c} = \frac{\lambda}{3.413\cdot\sin(60°)} = 0.338\lambda$$

b. $$D_0 = \frac{1.086\pi}{\sin^2\theta_c} = \frac{1.086\pi}{\sin^2(60°)} = 4.55 = 6.58\,\text{dB}$$

c. $$P(\theta=0)\big|_{max} = (2\pi a)^2$$

$$P(\theta)\Big|_{\theta=60°} = (2\pi a)^2\cdot\left\{\frac{2J_1(ka\sin\theta)}{ka\sin\theta}\right\}^2, \quad P(\theta=60°) = (2\pi a)^2\left\{\frac{2J_1(1.8392)}{1.8392}\right\}^2$$

$$P(\theta=60°) = (2\pi a)^2\cdot\left\{\frac{2(0.5819)}{1.8392}\right\}^2 = (2\pi a)^2\cdot(0.4004)$$

$$\left.\frac{P(\theta=\theta_c)}{P(\theta=0)}\right|_{\theta_c=60°} = 0.4004 = -3.975\,\text{dB}$$

12-46. a. $$a = \left.\frac{\lambda}{2.7318\cdot\sin(\theta_c)}\right|_{\theta_c=60°} = \frac{\lambda}{2.3658} = 0.42268\lambda$$

b. $$D_0 = \frac{1.263\pi}{\sin^2(\theta_c)} = \frac{1.263\cdot\pi}{0.75} = 5.2904 = 7.235\,\text{dB}$$

c. $$P(\theta) = 0.75(2\pi a)^2\cdot\left\{\frac{8J_2(ka\sin\theta)}{[ka\sin\theta]^2}\right\}^2.$$

$$P(\theta=0) = 0.75\cdot(2\pi a)^2\cdot\left\{8\cdot J_2(x)/x^2\right\}\Big|_{x=0}^2 = 0.75(2\pi a)^2\cdot 1$$

$$P(\theta=60°) = 0.75\cdot(2\pi a)^2\cdot\left\{\frac{8J_2\left(\frac{2\pi}{\lambda}\cdot 0.42268\lambda\cdot\sin 60°\right)}{\left[\frac{2\pi}{\lambda}\cdot 0.42268\lambda\cdot\sin 60°\right]^2}\right\}^2$$

$$= 0.75(2\pi a)^2\cdot\{8\cdot J_2(2.3)/(2.3)^2\}^2 = 0.75(2\pi a)^2\cdot(0.39182)$$

$$\left.\frac{P(\theta=60°)}{P(\theta=0°)}\right|_{\theta_c=60°} = 0.39182 = -4.069134\,\text{dB}$$

12-47. a. $a = \frac{\lambda}{3.06317 \cdot \sin\theta_c} = \frac{\lambda}{3.06317 \cdot (0.8660)} = 0.37696 \cdot \lambda$

b. To find the maximum directivity, we need to drive the far-field for a circular aperture with a parabolic taper on -10 dB pedestral.

$\underline{E}_o = \hat{a}_y E_o \left(1 - \left(\frac{\rho'}{1.2099 \cdot a}\right)^2\right) \Rightarrow \underline{M}_s = -2\hat{n} \times E_a = \hat{a}_x 2E_o\left(1 - \left(\rho'/1.2099a\right)^2\right)$

$L_\theta = 2E_o \cos\theta \cdot \cos\phi \int_0^{2\pi}\int_0^a \left(1 - \left(\frac{\rho'}{1.2099a}\right)^2\right) \cdot \rho' d\rho d\phi$

$L_\theta = 4\pi E_o \cos\theta \cos\phi \left[\int_0^a \rho' J_0(k\rho'\sin\theta) d\rho' - \frac{1}{(1.2099 \cdot a)^2}\int_0^a \rho'^3 J_0(k\rho'\sin\theta) d\rho'\right]$

Making a change of variable of the form. $X = k\rho'\sin\theta \Rightarrow dX = k\sin\theta d\rho'$

We can write it as

$L_\theta = 4\pi E_o \cos\theta \cos\phi \left\{\frac{1}{(k\sin\theta)^2}\int_0^{k a\sin\theta} X J_0(X) dX - \frac{1}{(1.2099 \cdot a)^2}\frac{1}{(k\sin\theta)^4}\left[\int_0^{k a\sin\theta} X^3 J_0(X) dX\right]\right\}$

$= 4\pi E_o \cos\theta \cdot \cos\phi \left[\frac{a^2 J(X)}{X} - \frac{a^2}{(1.2099)^2}\frac{J(X)}{X} + \frac{2a^2}{(1.2099)^2}\cdot\frac{J_2(X)}{X^2}\right]$

$\left(\leftarrow \int X J_0(X) dX = X J_1(X) + C, \int X^3 J_0(X) dX = X^3 J_1(X) - 2X^2 J_2(X) + C\right)$

As similiar procedure.

$L_\phi = -4\pi E_o \sin\phi \cdot a^2 \left[\left(1 - \frac{1}{(1.2099)^2}\right)\frac{J_1(X)}{X} + \frac{2}{(1.2099)^2}\cdot\frac{J_2(X)}{X^2}\right]$.

$\therefore E_\theta = -j\frac{ke^{-jkr}}{4\pi r} \cdot L_\phi = j\frac{kE_o \cdot a^2 e^{-jkr}}{r} \cdot \sin\phi \left[0.316647\frac{J_1(X)}{X} + 1.3667 \cdot \frac{J_2(X)}{X^2}\right]$

$E_\phi = j\frac{kE_o a^2 e^{-jkr}}{r} \cdot \cos\theta \cdot \cos\phi \left[0.316647\frac{J_1(X)}{X} + 1.3667\ \frac{J_2(X)}{X^2}\right]$.

Radiated Power reduces to

$P_{rad} = \oint\oint_S W_{av} dS = \frac{|E_o|^2}{2\eta} \cdot \int_0^{2\pi}\int_0^a \left(1 - \left(\frac{\rho'}{1.2099 \cdot a}\right)^2\right)^2 \cdot \rho' d\rho' d\phi$

$= \frac{|E_o|^2}{2\eta} \cdot 2\pi \cdot \left[\frac{a^2}{2} - \frac{1}{2}\frac{a^2}{(1.2099)^2} + \frac{1}{6}\frac{a^2}{(1.2099)^4}\right] = \frac{|E_o|^2}{2\eta} \cdot 2\pi \cdot 0.23615 \cdot a^2$

$U_{max} = \frac{|E_o|^2}{2\eta} \cdot a^4 \cdot \left(\frac{2\pi}{\lambda}\right)^2 \cdot \left[0.316647 \cdot \frac{J_1(X)}{X} + 1.3667 \cdot \frac{J_2(X)}{X^2}\right]^2\Big|_{X=0}$.

$= \frac{|E_o|^2}{2\eta} \cdot a^4 \cdot \left(\frac{2\pi}{\lambda}\right)^2 \cdot \left[(0.32916)^2\right]$ $\left(\leftarrow \frac{J_1(X)}{X}\Big|_{X=0} = 0.5,\ \leftarrow \frac{J_2(X)}{X^2}\Big|_{X=0} = \frac{1}{8}\right)$

$D_o = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi \cdot \frac{|E_o|^2}{2\eta} \cdot \left(\frac{2\pi}{\lambda}\right)^2 \cdot (0.1083469) \cdot a^4}{2\pi \cdot \frac{|E_o|^2}{2\eta} \cdot 0.23615 \cdot a^2} = 2 \cdot a^2 \cdot \frac{4\pi}{\lambda^2} \cdot \pi \cdot (0.458805)$

$= 0.91761 \cdot \frac{4\pi}{\lambda^2}(\pi a^2)$.

(Continued)

12-47. (Cont'd) For a circular aperture with parabolic taper with −10 dB pedestral, the normalized power pattern multiplied by the maximum directivity can be written as

$$P(\theta) = 0.91761 \cdot (2\pi a)^2 \cdot \left\{ (3.038026)^2 \left[ 0.316647 \frac{J_1(X)}{X} + 1.3667 \cdot \frac{J_2(X)}{X^2} \right]^2 \right\}$$

$\leftarrow X = ka\sin\theta$

For any other angle $\theta = \theta_c$, the maximum of the pattern occurs when

$ka\sin\theta_c = 2.0512$ $\rightarrow$ [$0.316647\, J_1(X) + 1.3667 \cdot \frac{J_2(X)}{X}$, has maximum]

$\therefore$ Optimum Radius $a = \frac{\lambda}{\frac{2\pi}{2.0512}\cdot\sin\theta_c} = \frac{\lambda}{3.06317\cdot\sin\theta_c}$

Directivity is

$$D = 0.91761 \cdot \frac{4\pi}{\lambda^2} \cdot \pi \cdot \left( \frac{\lambda}{3.06317\cdot\sin\theta_c} \right) = \frac{1.228247\cdot\pi}{(\sin\theta_c)^2}$$

The maximum directivity with aperture radius $a \left(= \frac{\lambda}{3.06317\cdot\sin 60^\circ}\right)$ is

$$D = \frac{1.228247\cdot\pi}{(\sin 60^\circ)^2} = 5.144869 = 7.1137\ \text{dB}.$$

c. The value of the directivity at the edge of the desired coverage ($\theta = \theta_c = 60^\circ$), relative to its maximum value $\theta = 0$, is

$$\frac{P(\theta=\theta_c)}{P(\theta=0)}.$$

$$\therefore P(\theta=0) = 0.91761 \cdot (2\pi a)^2.$$

$$ka\cdot\sin\theta_c\Big|_{\theta_c=60^\circ} = \frac{2\pi}{\lambda}\cdot\left(\frac{\lambda}{3.06317}\cdot\frac{1}{\sin\theta_c}\right)\cdot\sin\theta_c$$

$P(\theta=\theta_c) = P(\theta_c=60^\circ)$

($\leftarrow ka\sin\theta_c = 2.0512$.)

$$P(\theta_c=60^\circ) = 0.91761\cdot(2\pi a^2)\left\{(3.038026)^2\cdot\left[0.316647\,\frac{J_1(2.0512)}{(2.0512)} + 1.3667\cdot\frac{J_2(2.0512)}{(2.0512)^2}\right]^2\right\}$$

$$= 0.91761\cdot(2\pi a^2)\left[(3.038026)^2\cdot[0.316647\cdot(0.27930) + 1.3667\cdot(0.086548)]^2\right\}$$

$$= 0.91761\cdot(2\pi a^2)\cdot\left[(3.038026)\cdot(0.20672522)\right]^2$$

$$= 0.91761\,(2\pi a^2)\cdot[0.39443]$$

$$\frac{P(\theta=\theta_c)}{P(\theta=0)} = 0.39443 = -4.0402979\ \text{dB}.$$

12-48. For circular aperture with parabolic-taper illumination.

the power pattern (Relative power pattern times the antenna directivity), is given by

$$P(\theta) = 0.75\,(\pi 2a)^2 \left\{ \frac{8 J_2(ka\sin\theta)}{[ka\sin\theta]^2} \right\}^2,$$

where a is the radius of the aperture in wavelengths, and $J_2$ is the second-order bessel function.

The maximum value of $P(\theta)$, for a given $\theta$, occurs when the function $[J_2(x)/x]$ is a maximum. ie.

$$ka\sin\theta_c = 2.3$$

$$\frac{2\pi}{\lambda} \cdot a \cdot \sin\theta_c = 2.3 \Rightarrow a = \frac{\lambda}{\frac{2\pi}{2.3}\sin\theta_c} = \frac{\lambda}{2.7318 \cdot \sin\theta_c}$$

$$D_o \text{ (directivity)} = 0.75\,\frac{4\pi}{\lambda^2}(\pi a^2)$$

$$= 0.75 \cdot \frac{4\pi^2}{\lambda^2} \cdot \left[\frac{\lambda}{2.7318 \cdot \sin(\theta_c)}\right]^2 = 0.75\,\frac{(4\pi)}{(2.7318)^2} \cdot \frac{\pi}{(\sin(\theta_c))^2}$$

$$D_o = 0.75 \cdot (1.68388)\,\frac{\pi}{[\sin(\theta_c)]^2}$$

$$= 1.2629\,\pi / \sin^2\theta_c$$

E.O.C. (Relative power at $\theta_c$ to maximum power)

$$\frac{P(\theta=\theta_c)}{P(\theta=0)} = \frac{0.75 \cdot (\pi \cdot 2a)^2 \left[8 \cdot \frac{J_2(2.3)}{(2.3)^2}\right]^2}{0.75 \cdot (\pi 2a)^2}$$

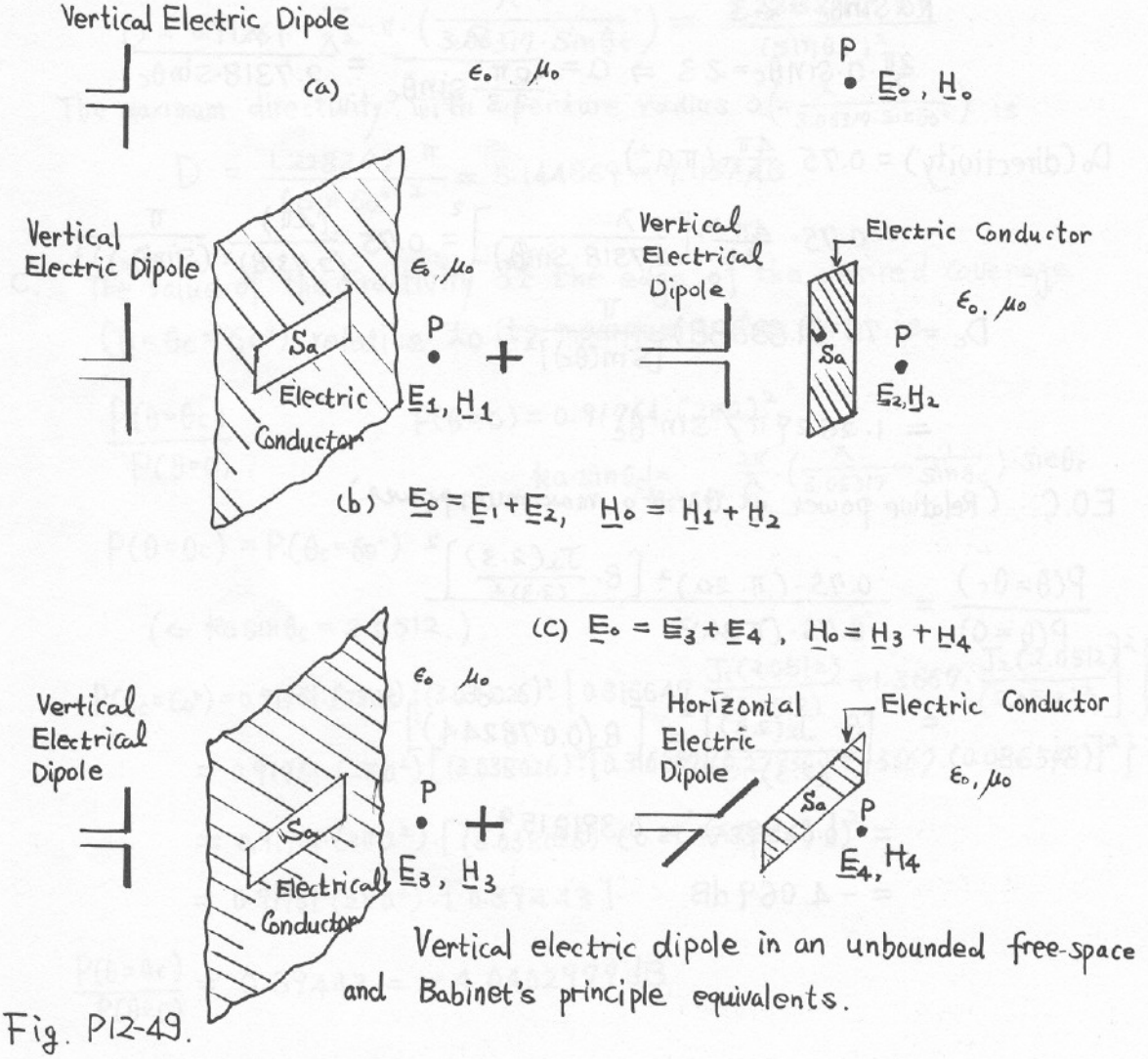
$$= \left[\frac{8 \cdot J_2(2.3)}{(2.3)}\right]^2 = \left[8 \cdot (0.078244)\right]^2$$

$$= (0.625952)^2 = 0.3918159$$

$$= -4.069\ \text{dB}$$

12-49. a. One method that can be used combines the results of a vertical dipole in the presence of a thin, plane, infinite, perfectly conducting electric screen with a horizontal opening and those of a vertical dipole in the presence of a flat, thin, perfectly conducting electrical vertical strip as shown in part b of the figure. The strip has been rotated to represent the magnetic equivalent of the screen's opening.

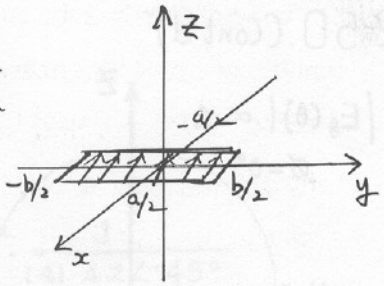
b. Another method combines the results of a vertical dipole in the presence of an electric conducting screen with a horizontal dipole in the presence of a horizontal electric conducting strip, as shown in part c of the figure. The dipole has been rotated to interchange the $\underline{E}$ and $\underline{H}$-fields and obtain the magnetic equivalent of the actual source.




Vertical electric dipole in an unbounded free-space and Babinet's principle equivalents.

Fig. P12-49.

12-50 $\vec{E} = -E_0 \cdot \hat{a}_x$ $\begin{pmatrix} -b/2 < y < b/2 \\ -a/2 < x < a/2 \end{pmatrix}$ $\begin{pmatrix} a = 0.1\lambda \\ b = 0.5\lambda \end{pmatrix}$




$\vec{E} = -\hat{a}_x E_0$

$\vec{M} = -\hat{n} \times 2\vec{E} = -\hat{n} \times (-2\hat{a}_x E_0) = \hat{a}_y 2E_0$

$$E_\theta = -C \cdot \cos\phi \frac{\sin X}{X} \cdot \frac{\sin Y}{Y} \quad , X = \frac{ka}{2}\sin\theta \cdot \cos\phi$$

$$E_\phi = C \cos\theta \cdot \sin\phi \frac{\sin X}{X} \cdot \frac{\sin Y}{Y} \quad , Y = \frac{kb}{2}\sin\theta \cdot \sin\phi$$

$$C = j\frac{ab k E_0 e^{-jkr}}{2\pi r}$$

a. $|E_\theta(\theta)|_{\substack{(x-z\ plane) \\ At\ \phi=0^\circ}} \propto \left|\frac{\sin X}{X}\right| \approx 1.$

$X = 0.1\cdot\pi\cdot\sin\theta\cos 0^\circ = 0.1\pi\sin\theta \rightarrow \frac{\sin(0.1\pi\sin\theta)}{(0.1\pi\sin\theta)} \approx 1.$

$Y = 0. \rightarrow \frac{\sin Y}{Y} = 1$

b. At $\phi = 90^\circ$ ($y-z$ plane)

$|E_\theta(\theta)|_{At\ \phi=90^\circ} \propto 0 \rightarrow$ zero

$X = 0 \rightarrow \sin X/X = 1$

$Y = \frac{\pi}{2}\sin\theta \rightarrow \sin Y/Y$

c. At $\theta = 90^\circ$ ($x-y$ plane)

$$|E_\theta(\phi)|_{At\ \theta=90^\circ} \propto \left|\cos\phi \cdot \frac{\sin X}{X} \cdot \frac{\sin Y}{Y}\right|$$

$X = 0.1\cdot\pi\cdot\cos\phi$

$Y = \frac{\pi}{2}\cdot\sin\phi$

d. At $\phi = 0^\circ$ ($x-z$ plane)

$|E_\phi(\theta)|_{At\ \phi=0^\circ} \propto |0|$, $\rightarrow$ zero

e. At $\phi = 90^\circ$, ($y-z$ plane)

$$|E_\phi(\theta)|_{At\ \phi=90^\circ} \propto \left|\cos\theta \cdot \frac{\sin Y}{Y}\right|$$

$Y = \frac{\pi}{2}\sin\theta$.

f. At $\theta = 90^\circ$ ($x-z$ plane)

$|E_\phi(\phi)|_{At\ \theta=90^\circ} \propto |0| \rightarrow$ zero (Continued)

12-50. (Cont'd)

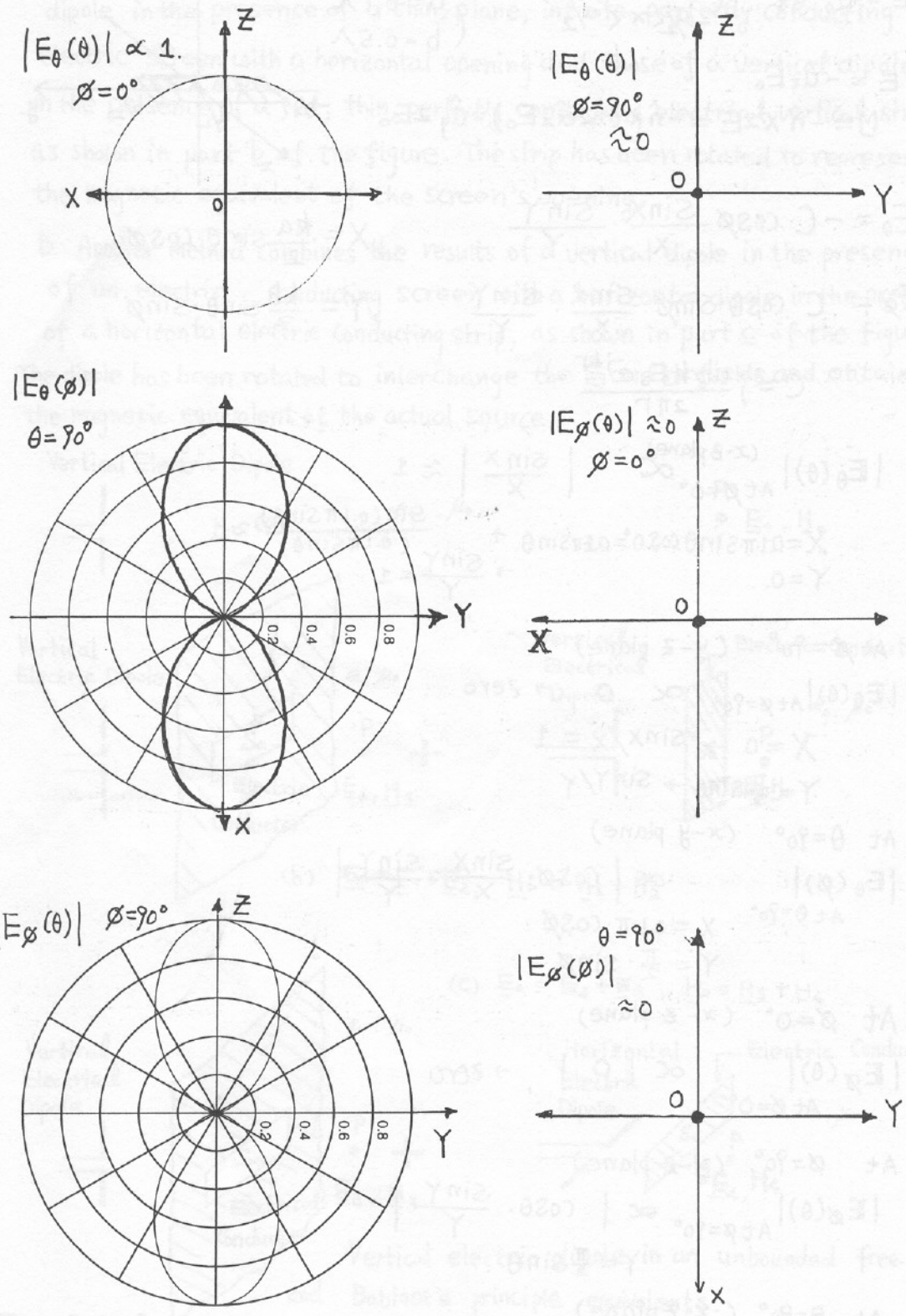



Fig. P 12-50

12-51. Referring to Fig. 5-11

$Z_{loop} = 100 - j100$

According to Babinet's Principle

$$Z_c \cdot Z_s = Z_{slot} \cdot Z_{loop} = \frac{\eta_0^2}{4}$$

$$Z_{slot} = \frac{\eta_0^2}{4} \cdot \frac{1}{Z_{loop}} = \left(\frac{(377)^2}{4}\right) \cdot \frac{1}{100 - j100} = \frac{(377)^2}{4} \cdot \frac{1}{141.42\angle -45^\circ}$$

$$= 251.27\angle 45^\circ$$

$$Z_{slot} = 177.65 + j177.65$$

12-52. $\underline{E}_a = \hat{a}_y E_0 \cos\left(\frac{\pi}{a}x\right)$

From (12-113a) and (12-113b)

$f_x = 0$

$$f_y = E_0 \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \cos\left(\frac{\pi}{a}x'\right) e^{jk(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)}\, dx'dy'$$

$$= E_0 \int_{-b/2}^{b/2} e^{jky'\sin\theta\sin\phi} \int_{-a/2}^{a/2} \cos\left(\frac{\pi}{a}x'\right) e^{jk(x'\sin\theta\cos\phi)}\, dx'dy'$$

$$\left(\leftarrow \int \cos Ax \cdot e^{Bx}dx = \frac{e^{Bx}(B\cos Ax + A\sin Ax)}{A^2 + B^2}\right)$$

$$\left(\therefore \leftarrow \int_{-a/2}^{a/2} \cos\frac{\pi}{a}x' \cdot e^{jkx'\sin\theta\cos\phi}dx' = -\frac{\pi}{2}a \cdot \frac{\cos X}{X^2 - \left(\frac{\pi}{2}\right)^2}, \quad X = \frac{ka\sin\theta\cos\phi}{2}\right)$$

$$f_y = E_0\left(-\frac{\pi}{2}ab\right) \cdot \frac{\cos X}{X^2 - \left(\frac{\pi}{2}\right)^2} \cdot \frac{\sin Y}{Y}, \qquad X = \frac{ka\sin\theta\cos\phi}{2}, \quad Y = \frac{kb\sin\theta\sin\phi}{2}$$

The $\hat{a}_\theta$ and $\hat{a}_\phi$ component of E-field can be written

$$E_\theta = -j\frac{\frac{\pi}{2}\cdot ab E_0}{2\pi r} k e^{-jkr} \sin\phi \cdot \frac{\cos X}{X^2 - \left(\frac{\pi}{2}\right)^2} \cdot \frac{\sin Y}{Y} = -j\frac{a\cdot bE_0}{4r} k e^{-jkr}\sin\phi \cdot \frac{\cos X}{X^2 - \left(\frac{\pi}{2}\right)^2} \cdot \frac{\sin Y}{Y}$$

$$E_\phi = -j\frac{\frac{\pi}{2}abE_0}{2\pi r} k e^{-jkr\cos\theta\cos\phi} \cdot \frac{\cos X}{X^2 - \left(\frac{\pi}{2}\right)^2} \cdot \frac{\sin Y}{Y} = -j\frac{a\cdot bE_0}{4r} k e^{-jkr\cos\theta\cos\phi} \cdot \frac{\cos X}{X^2 - \left(\frac{\pi}{2}\right)^2} \cdot \frac{\sin Y}{Y}$$

13-1. To find the fields within the horn, we will use the cylindrical coordinate system of $\rho, \psi, X$ of Fig 13.2(a). Since the region within the horn is source-free, Maxwell's equations

$$\nabla \times \underline{E} = -j\omega\mu \underline{H} \quad (1)$$

$$\nabla \times \underline{H} = j\omega\epsilon \underline{E} \quad (2)$$

must be satisfied. In terms of the cylindrical coordinates $\rho, \psi, X$, equations 1 and 2 reduces to

$$j\omega\epsilon E_\rho = \frac{1}{\rho}\frac{\partial H_X}{\partial \psi} - \frac{\partial H_\psi}{\partial X} \quad (3)$$

$$j\omega\epsilon E_\psi = \frac{\partial H_\rho}{\partial X} - \frac{\partial H_X}{\partial \rho} \quad (4)$$

$$j\omega\epsilon E_X = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\psi) - \frac{1}{\rho}\frac{\partial H_\rho}{\partial \psi} \quad (5)$$

$$-j\omega\mu H_\rho = \frac{1}{\rho}\frac{\partial E_X}{\partial \psi} - \frac{\partial E_X}{\partial X} \quad (6)$$

$$-j\omega\mu H_\psi = \frac{\partial E_\rho}{\partial X} - \frac{\partial E_X}{\partial \rho} \quad (7)$$

$$-j\omega\mu H_X = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\psi) - \frac{1}{\rho}\frac{\partial E_\rho}{\partial \psi} \quad (8)$$

If we assume that the waveguide feeding the horn only supports the dominant $TE_{10}$-mode, the lowest order mode within the sectoral guide (horn) is that which is analogous to the $TE_{10}$-mode of the rectangular guide, with all the other modes attenuated in the transition region (throat) between the waveguide and the horn. Thus the dominant mode within the horn is one whose only non vanishing components are $E_\psi, H_\rho, H_X$.

That is $E_\rho = E_X = H_\psi = 0$ (9)

In addition $\frac{\partial}{\partial \psi} = 0$ (10)

Using (9) and (10), we can write (6) and (8) as

$$j\omega\mu H_\rho = \frac{\partial E_\psi}{\partial X} \quad (11)$$

$$-j\omega\mu H_X = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\psi) \quad (12)$$

Substituting (11) and (12) into (4) we can write

$$-\omega^2\mu\epsilon E_\psi = \frac{\partial^2 E_\psi}{\partial X^2} + \frac{\partial}{\partial \rho}\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\psi)\right] \quad (13)$$

which when expanded can be written as

(Continued)

13-1 (Cont'd)

$$\frac{\partial^2 E_\psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_\psi}{\partial \rho} + \frac{\partial^2 E_\psi}{\partial x^2} + \left(k^2 - \frac{1}{\rho^2}\right) E_\psi = 0 \qquad (14)$$

where $k^2 = \omega^2 \mu \varepsilon$ (14a)

To solve (14), we make use of the method of separation of variables. We assume that

$$E_\psi(\rho, x) = R(\rho) X(x) \qquad (15)$$

Substituting (15) into (14) leads to

$$X\frac{\partial^2 R}{\partial \rho^2} + X\frac{1}{\rho}\frac{\partial R}{\partial \rho} + R\frac{\partial^2 X}{\partial x^2} + \left(k^2 - \frac{1}{\rho^2}\right) RX = 0 \qquad (16)$$

Dividing by RX and changing the partials to total derivation (16) reduces to

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -k_x^2 \quad , \quad k_x^2 = \text{constant} \qquad (17)$$

$$\frac{1}{R}\frac{d^2 R}{d\rho^2} + \frac{1}{R}\frac{1}{\rho}\frac{\partial R}{\partial \rho} + \left[\left(k^2 - k_x^2\right) - \frac{1}{\rho^2}\right] = 0 \qquad (18)$$

Multiplying (18) by $\rho^2 R$ reduces to

$$\rho^2 \frac{d^2 R}{d\rho^2} + \rho \frac{dR}{d\rho} + \left[(k_\rho \rho)^2 - 1\right] R = 0 \qquad (19)$$

where $k_\rho^2 = k^2 - k_x^2$ (19a)

Equation (19) is recognized as a special form ($n=1$) of Bessel's differential equation [equation V-1 of Appendix V] with a solution of

$$R(\rho) = A H_1^{(2)}(k_\rho \rho) + B_1 H_1^{(1)}(k_\rho \rho) \qquad (20)$$

where A and B are constants. The Hankel functions of the first and second kind of order one ($n=1$) were chosen as solutions because they represent traveling waves in the inward and outward, respectively, radial directions. The solution of (17) is of the form

$$X(x) = C\cos(k_x x) + D\sin(k_x x) \qquad (21)$$

where C and D are constants. Using (20) and (21) we can write (15) as

$$E_\psi(\rho, x) = \left[A H_1^{(2)}(k_\rho \rho) + B_1 H_1^{(2)}(k_\rho \rho)\right]\left[C\cdot\cos(k_x x) + D\cdot\sin(k_x x)\right] \qquad (22)$$

(Continued)

13-1. (Cont'd) Applying the boundary conditions

$$E_\psi(\rho, x=a/2) = E_\psi(\rho, x=-a/2) = 0 \tag{23}$$

leads to

$$\left.\begin{aligned} C\cos\left(\frac{k_x a}{2}\right) + D\sin\left(\frac{k_x a}{2}\right) &= 0 \\ C\cos\left(\frac{k_x a}{2}\right) - D\sin\left(\frac{k_x a}{2}\right) &= 0 \end{aligned}\right\} \Rightarrow D=0,\ k_x = m\left(\frac{\pi}{a}\right) \tag{24}$$

$$m = 1, 2, 3, \cdots \tag{25}$$

Equation (22) can be rewritten as

$$E_\psi(\rho, x) = A_m \cos(k_x x)\left[H_1^{(2)}(k_\rho \rho) + \alpha_m H_1^{(1)}(k_\rho \rho)\right] \tag{26}$$

where $A_m$ and $\alpha_m$ are constants for the modes $m = 1, 3, 5, \cdots$

The nonvanishing magnetic field components can be obtained from (11) and (12). That is

$$H_\rho(\rho, x) = \frac{1}{j\omega\mu}\frac{\partial E_\psi}{\partial x} = j\frac{k_x}{\omega\mu} A_m \sin(k_x x)\left[H_1^{(2)}(k_\rho \rho) + \alpha_m H_1^{(1)}(k_\rho \rho)\right] \tag{27}$$

$$H_x(\rho, x) = -\frac{1}{j\omega\mu}\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\psi) = j\frac{k_\rho}{\omega\mu} A_m \cos(k_x x)\left[H_0^{(2)}(k_\rho \rho) + \alpha_m H_0^{(1)}(k_\rho \rho)\right] \tag{28}$$

by using (V-18).

If we consider only the lowest order mode ($m=1$, $k_x = \pi/a$) and no reflected component [$\alpha_m H_1^{(1)}(k_\rho \rho) = \alpha_m H_0^{(1)}(k_\rho \rho) = 0$], the fields within the horn can be written as

$$E_\rho = E_x = H_\psi = 0 \tag{29a}$$

$$E_\psi(\rho, x) = A_1 \cos\left(\frac{\pi}{a}x\right) H_1^{(2)}(k_\rho \rho) \tag{29b}$$

$$H_\rho(\rho, x) = j\frac{\pi}{\omega\mu a} A_1 \sin\left(\frac{\pi}{a}x\right) H_1^{(2)}(k_\rho \rho) \tag{29c}$$

$$H_x(\rho, x) = j\frac{k_\rho}{\omega\mu} A_1 \cos\left(\frac{\pi}{a}x\right) H_0^{(2)}(k_\rho \rho) \tag{29d}$$

$$k_\rho = \left[k^2 - (\pi/a)^2\right]^{1/2} = k\left[1 - \left(\frac{\pi}{ka}\right)^2\right]^{1/2} = k\left[1 - (f_c/f)^2\right]^{1/2} \overset{f \gg f_c}{\simeq} k \tag{29e}$$

$$\omega\mu/k_\rho \simeq \omega\mu/k = \eta \tag{29f}$$

The cylindrical components $E_\psi(\rho, x)$ and $H_\rho(\rho, x)$ can be resolved to any point within the horn to their rectangular counterparts using (VII-7a) or (VII-7b). Thus

$$E_z = -E_\psi \sin\psi, \qquad H_z = H_\rho \cos\psi \tag{30}$$

$$E_y = E_\psi \cos\psi, \qquad H_y = H_\rho \sin\psi \tag{31}$$

(Continued)

13-1 (Cont'd)

which for horns with small flares ($\psi$ small) reduce to

$$E_z = H_y = 0 \tag{32}$$

$$E_y \simeq E_\psi, \quad H_z \simeq H_\rho \tag{33}$$

If the length of the horn is large, the Hankel functions can be approximated by their asymptotic expansions of (V-17), or

$$H_1^{(2)}(k\rho) \simeq j\sqrt{\frac{2}{\pi k\rho}}\, e^{-j(k\rho-\pi/4)}, \quad H_0^{(2)}(k\rho) \simeq \sqrt{\frac{2}{\pi k\rho}}\, e^{-j(k\rho-\pi/4)} \tag{34}$$

where $$\rho = (y^2+z^2)^{1/2} \tag{34a}$$

Choosing a new coordinate system $(x', y', z')$, as shown in Fig. 13.2(b) such that

$$x' = x, \; y' = y, \; z' = z - \rho_1, \; \rho_1 = \rho\cos\psi_e \tag{35}$$

we can write $\rho$ of (34a) as

$$\rho = (z^2+\rho^2)^{1/2} = [(z'+\rho_1)^2+y'^2]^{1/2} \overset{z'=0}{=} [\rho_1^2+y'^2]^{1/2} = \rho_1\left[1+\frac{1}{2}\left(\frac{y'}{\rho_1}\right)^2 - \frac{1}{8}\left(\frac{y'}{\rho_1}\right)^4+\cdots\right] \tag{36}$$

For narrow horns ($y' \ll \rho_1$)

$$\rho \simeq \begin{cases} \rho_1\left[1+\frac{1}{2}\left(\frac{y'}{\rho_1}\right)^2\right] & \text{for phase terms} \\ \rho_1 & \text{for amplitude terms} \end{cases} \tag{37}$$

Using (30)-(37), we can write (29a)-(29b) as

$$E_z' = E_x' = H_y' = 0$$

$$E_1 = j A_1 \sqrt{\frac{2}{\pi k\rho_1}}\, e^{-jk\rho_1},$$

$$E_y'(x', y') \simeq E_1 \cos\left(\frac{\pi}{a}x'\right) e^{-j\frac{k}{2}\left(\frac{y'}{\rho_1}\right)^2}$$

$$H_z'(x', y') \simeq j E_1\left(\frac{\pi\eta}{ka}\right)\sin\left(\frac{\pi}{a}x'\right) e^{-j\frac{k}{2}\left(\frac{y'}{\rho_1}\right)^2}$$

$$H_x'(x', y') \simeq +\frac{E_1}{\eta}\cos\left(\frac{\pi}{a}x'\right) e^{-j\frac{k}{2}\left(\frac{y'}{\rho_1}\right)^2}$$

13-2. Using the geometry of Figure 13.2(b)

a. $\Phi_{max} = \frac{2\pi}{3} = k\delta_{max} = \frac{2\pi}{\lambda}(\rho_e - \rho_1) = \frac{2\pi}{\lambda}(\rho_e - \rho_e\cos\psi_e) = \frac{2\pi}{\lambda}\rho_e(1-\cos\psi_e)$

$\frac{2\pi}{3} = \frac{2\pi}{\lambda}(10\lambda)(1-\cos\psi_e) = 2\pi(10\lambda)(1-\cos\psi_e) = 2\pi(10)\left[2\sin^2\left(\frac{\psi_e}{2}\right)\right]$

$\sin^2\frac{\psi_e}{2} = \frac{1}{3(20)} = \frac{1}{60} \Rightarrow \frac{\psi_e}{2} = \sin^{-1}\left(\frac{1}{\sqrt{20}}\right) = 7.418°$

$\psi_e = 2(7.418°) = 14.836°, \quad 2\psi_e = 29.672°$

b. $\frac{b_1}{2} = \rho_e\sin\psi_e = 10\lambda\sin(14.836°) = 2.56\lambda \Rightarrow b_1 = 5.12\lambda$

c. $\rho_1 = \rho_e\cos\psi_e = 10\lambda\cos(14.836°) = 9.667\lambda$

$$D_E = \frac{64a\rho_1}{\pi\lambda b_1}\left[C^2\left(\frac{b_1}{\sqrt{2\lambda\rho_1}}\right) + S^2\left(\frac{b_1}{\sqrt{2\lambda\rho_1}}\right)\right]$$

$$\frac{b_1}{\sqrt{2\lambda\rho_1}} = \frac{5.12\lambda}{2\lambda(9.667\lambda)} = 1.1644$$

$$C^2\left(\frac{b_1}{\sqrt{2\lambda\rho_1}}\right) = [C(1.1644)]^2 = (0.738)^2 = 0.5446$$

$$S^2\left(\frac{b_1}{\sqrt{2\lambda\rho_1}}\right) = [S(1.1644)]^2 = (0.591)^2 = 0.3493$$

$$D_E = \frac{64(0.5\lambda)(9.667\lambda)}{\pi(5.12)\lambda^2}(0.5446 + 0.3493) = 17.192 = 12.35\,dB$$

d. $G_E = \epsilon_t D_E = \epsilon_r \epsilon_{cd} D_E = \epsilon_r(1) D_E = (1-|\Gamma|^2) D_E$

$= [1-(0.2)^2](17.192) = 0.96(17.192) = 16.50$

$= 12.18\,dB$

13-3. $\rho_1 = 6\lambda,\ b_1 = 3.47\lambda,\ a = 0.5\lambda \Rightarrow \rho_e = [\rho_1^2 + (b_1/2)^2]^{1/2} = 6.2458\lambda$

a. $S = \dfrac{b_1^2}{8\lambda\rho_1} = \dfrac{(3.47\lambda)^2}{8\lambda \cdot 6\lambda} = 0.25$

$\theta = 0°$ : $\dfrac{b_1}{\lambda}\sin\theta = 0$

Using Fig. 13.6 we can write, that for $S = 1/4$, at $\theta = 0°$.

$E = 0. + 20\log_{10}\left[\dfrac{1+\cos(0°)}{2}\right] = 0 + 0 = 0\,dB$

$\theta = 10°$ : $\dfrac{b_1}{\lambda}\sin\theta = 3.47\sin(10°) = 0.6$. Thus at $\theta = 10°$

$E = -3.5 + 20\log_{10}\left[\dfrac{1+\cos(10°)}{2}\right] = -3.5 - 0.066 = -3.566\,dB$

$\theta = 20°$ : $\dfrac{b_1}{\lambda}\sin\theta = 3.47\sin(20°) = 1.187$

$E = -9.25 + 20\log_{10}\left[\dfrac{1+\cos(20°)}{2}\right] = -9.25 - 0.266 = -9.516\,dB$

b. $D_E = \dfrac{64\,\rho_1 a}{\pi b_1 \lambda}\left[C^2\left(\dfrac{b_1}{\sqrt{2\lambda\rho_1}}\right) + S^2\left(\dfrac{b_1}{\sqrt{2\lambda\rho_1}}\right)\right]$ (13-19)

$\dfrac{b_1}{\sqrt{2\lambda\rho_1}} = \dfrac{3.47\lambda}{\sqrt{2\lambda \cdot 6\lambda}} = 1.00$

$C^2(1.00) = (0.77989)^2 = 0.6082$

$S^2(1.00) = (0.43826)^2 = 0.1921$

$D_E = \dfrac{64(6\lambda)(0.5\lambda)}{\pi(3.47\lambda)(\lambda)}\left[0.6082 + 0.1921\right] = 14.095 = 11.49\,dB$

$B = \dfrac{b_1}{\lambda}\sqrt{\dfrac{50}{\rho_e/\lambda}} = 3.47\sqrt{\dfrac{50}{6.2458}} = 9.8179$

From Fig. 13.9 $\Rightarrow G_E = 81.32$

Using (13-20c)

$D_E = \dfrac{a}{\lambda}\dfrac{G_E}{\sqrt{\frac{50}{\rho_e/\lambda}}} = 0.5\,\dfrac{81.32}{\sqrt{\frac{50}{6.2458}}} = 14.37 = 11.57\,dB$

13-4. $\rho_1 = 6\lambda,\ b_1 = 6\lambda,\ a = 0.5\lambda \Rightarrow \rho_e = [\rho_1^2 + (b_1/2)^2]^{1/2} = (6^2+3^2)^{1/2}\lambda$

$\rho_e = 6.708\lambda$

a. $S = b_1^2/8\lambda\rho_1 = 6^2/8(6) = 3/4$

$\theta = 0°$: $\frac{b_1}{\lambda}\sin\theta = 0$. From Fig. 13.6 $\Rightarrow E = -3.3 + 20\log_{10}\left(\frac{1+\cos\theta}{2}\right)_{\theta=0°} = -3.3 + 0 = -3.3$dB

$\theta = 10°$: $\frac{b_1}{\lambda}\sin\theta\Big|_{\theta=10°} = 1.042 \Rightarrow E = -0.25 + 20\log_{10}\left(\frac{1+\cos\theta}{2}\right)_{\theta=10°} = -0.25 - 0.066 = -0.316$ dB

$\theta = 20°$: $\frac{b_1}{\lambda}\sin(20°) = 2.052 \Rightarrow E = -3.25 + 20\log_{10}\left(\frac{1+\cos\theta}{2}\right)_{\theta=20°} = -3.25 - 0.266 = -3.516$dB

b. $\frac{b_1}{\sqrt{2\lambda\rho_1}} = \frac{6}{\sqrt{12}} = 1.732$

Using Tables in Appendix IV $\Rightarrow C^2(1.732) = (0.327)^2 = 0.10693$

$S^2(1.732) = (0.51776)^2 = 0.2681$

Using (13-19) $\Rightarrow D_E = \frac{64a\rho_1}{\pi b_1 \lambda}\left[C^2\left(\frac{b_1}{\sqrt{2\lambda\rho_1}}\right) + S^2\left(\frac{b_1}{\sqrt{2\lambda\rho_1}}\right)\right]$

$D_E = \frac{64(0.5)6}{\pi(6)}\left[0.10693 + 0.2681\right] = 3.8187 = 5.82$ dB

$B = \frac{b_1}{\sqrt{\lambda}}\sqrt{\frac{50}{\rho_e/\lambda}} = 6\sqrt{\frac{50}{6.708}} = 16.38$

From Fig. 13.9 $\Rightarrow G_E = 30.316$

Using (13-20c) $\Rightarrow D_E = \frac{a}{\lambda}\frac{G_E}{\sqrt{\frac{50}{\rho_e/\lambda}}} = \frac{0.5(30.316)}{\sqrt{50/6.708}} = 5.552 = 7.444$ dB

13-5. $b_1 = \sqrt{2\lambda\rho_1}$

| Using (13-19a) | Using Figure 13.8 |
|---|---|
| $\rho_1 = 6\lambda \Rightarrow b_1 = 3.46\lambda$ | $b_1 \simeq 3.5\lambda$ |
| $\rho_1 = 10\lambda \Rightarrow b_1 = 4.47\lambda$ | $b_1 \simeq 4.5\lambda$ |
| $\rho_1 = 15\lambda \Rightarrow b_1 = 5.48\lambda$ | $b_1 \simeq 5.5\lambda$ |
| $\rho_1 = 20\lambda \Rightarrow b_1 = 6.32\lambda$ | $b_1 \simeq 6.5\lambda$ |
| $\rho_1 = 30\lambda \Rightarrow b_1 = 7.75\lambda$ | $b_1 \simeq 8.0\lambda$ |
| $\rho_1 = 50\lambda \Rightarrow b_1 = 10.0\lambda$ | $b_1 \simeq 10.3\lambda$ |
| $\rho_1 = 75\lambda \Rightarrow b_1 = 12.25\lambda$ | $b_1 \simeq 12.5\lambda$ |
| $\rho_1 = 100\lambda \Rightarrow b_1 = 14.14\lambda$ | $b_1 \simeq 14.5\lambda$ |

13-6. $\rho_1 = 20\lambda$, $a = 0.5\lambda$

a. From Figure 13.8 $\Rightarrow b_1 = 6.5\lambda$

b. $\tan\psi_e = \frac{b_1/2}{\rho_1} = \frac{6.5\lambda/2}{20\lambda} = \frac{6.5}{40} \Rightarrow \psi_e = \tan^{-1}\left(\frac{6.5}{40}\right) = \tan^{-1}(0.1625) = 9.23°$

$2\psi_e = 2(9.23°) = 18.46°$

c. $D_E = \frac{64\rho_1 a}{\pi b_1 \lambda}\left[C^2\left(\frac{b_1}{\sqrt{2\lambda\rho_1}}\right) + S^2\left(\frac{b_1}{\sqrt{2\lambda\rho_1}}\right)\right]$ (13-19)

$\frac{b_1}{\sqrt{2\lambda\rho_1}} = \frac{6.5\lambda}{\sqrt{2\lambda(20\lambda)}} = 1.028$

Using the table in Appendix IV

$C^2(1.028) = (0.77)^2 = 0.5929$

$S^2(1.028) = (0.44)^2 = 0.1936$

$D_E = \frac{64(20)(0.5)}{\pi(6.5)}(0.5929 + 0.1936) = 24.65 = 13.92\text{dB}$

which compares very well with the value from Fig. 13.8

d. From Figure 13.7 $\Rightarrow$ HPBW $\simeq 10°$

e. $\rho_e = [\rho_1^2 + (b_1/2)^2]^{1/2} \simeq [(20)^2 + (6.5/2)^2]^{1/2}\lambda = 20.26\lambda$

$B = \frac{b_1}{\lambda}\sqrt{\frac{50}{\rho_e/\lambda}} = 6.5\sqrt{\frac{50}{20.26}} = 10.21 \Rightarrow G_E = 81.6$ (from Figure 13.9)

Using (13-20c) $\Rightarrow D_E = \frac{a}{\lambda}\sqrt{\frac{G_E}{50/\rho_e/\lambda}} = \frac{0.5(81.6)}{\sqrt{50/20.26}} = 25.917 = 14.14\text{dB}$

13-7. $a = 0.9\text{ in} = 2.286\text{ cm} \Rightarrow a = \frac{2.286}{2.7273}\lambda = 0.8382\lambda$

$b = 0.4\text{ in} = 1.016\text{ cm} \Rightarrow b = \frac{1.016}{2.7273}\lambda = 0.3725\lambda$

$\lambda = \frac{30\times10^9}{11\times10^9} = 2.7273\text{ cm}$

$D_E = 30 = 14.77\text{dB} \Rightarrow \frac{D_E\lambda}{a} = \frac{30\lambda}{0.8382\lambda} = 35.79 \simeq 36$

Using Figure 13.8 $\Rightarrow \rho_1 = 10\lambda$, $b_1 \simeq 4.5\lambda$, for $\frac{\lambda D_E}{a} \simeq 36$

$\psi_e = \tan^{-1}\left(\frac{b_1/2}{\rho_1}\right) = \tan^{-1}(0.225) = 12.68°$

$2\psi_e = 25.36°$

13-8. a. $b_1 \simeq \sqrt{2\lambda\rho_1} = \sqrt{2\cdot\lambda\cdot(10\lambda)} = \sqrt{20\lambda^2} = 4.4721\lambda$

$\rho_e = \sqrt{\rho_1^2 + (b_1/2)^2} = \sqrt{(10)^2 + (4.4721/2)^2}\cdot\lambda = 10.2470\lambda$

$\psi_e = \tan^{-1}\left(\frac{b_1/2}{\rho_1}\right) = \tan^{-1}\left(\frac{2.2361\lambda}{10\lambda}\right) = 12.6044^\circ = 0.2200 \text{ (rads)}$

b. $B = \frac{b_1}{\lambda}\sqrt{\frac{50}{\rho_e/\lambda}} = 4.4721\cdot\sqrt{\frac{50}{10.2470}} = 4.4721\sqrt{4.8795} = 4.4721(2.2090)$

$B = 9.8787 \Rightarrow$ From Figure 13.9 $\Rightarrow G_E \simeq 81.5$

$D_E = \frac{a}{\lambda}\frac{G_E}{\sqrt{\frac{50}{\rho_e/\lambda}}} = 0.7620\frac{81.5}{\sqrt{\frac{50}{10.2470}}} = 0.7620\frac{81.5}{2.2090} = 28.114 = 14.49\text{dB}$

$\lambda = \frac{c}{f} = \frac{30\times10^9}{10\times10^9} = 3\text{cm} = 1.1811''$

$a = \frac{0.9}{1.1811}\lambda = 0.7620\lambda,\quad b = \frac{0.4}{1.1811}\lambda = 0.3387\lambda$

c. $\epsilon_{ap} = \frac{A_{em}}{A_p} = \frac{2.237\lambda^2}{0.7620\lambda(4.4721\lambda)} = 0.6565 = 65.65\%$

$A_{em} = \frac{\lambda^2}{4\pi}G_0 \simeq \frac{\lambda^2}{4\pi}D_0 = \frac{\lambda^2}{4\pi}(28.114) = 2.2372\lambda^2$

d. $\Delta\phi_{max} = k(\rho_e - \rho_1) = \frac{2\pi}{\lambda}(10.2470 - 10)\lambda = 1.5519 \text{ rads} = 88.93^\circ$

13-9. To find the fields within an H-plane sectoral horn we can use a procedure similar to that of an E-plane sectoral horn of Problem 13.1. For the H-plane horn, Maxwell's equations reduce to

$$j\omega\epsilon E_\rho = \frac{1}{\rho}\frac{\partial H_y}{\partial\psi} - \frac{\partial H_\psi}{\partial y} \quad (1)$$

$$j\omega\epsilon E_\psi = \frac{\partial H_\rho}{\partial y} - \frac{\partial H_y}{\partial\rho} \quad (2)$$

$$j\omega\epsilon E_y = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\psi) - \frac{1}{\rho}\frac{\partial H_\rho}{\partial\psi} \quad (3)$$

$$-j\omega\mu H_\rho = \frac{1}{\rho}\frac{\partial E_y}{\partial\psi} - \frac{\partial E_\psi}{\partial y} \quad (4)$$

$$-j\omega\mu H_\psi = \frac{\partial E_\rho}{\partial y} - \frac{\partial E_y}{\partial\rho} \quad (5)$$

$$-j\omega\mu H_y = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\psi) - \frac{1}{\rho}\frac{\partial E_\rho}{\partial\psi} \quad (6)$$

Using the geometry of Figure 13.10. The nonvanishing components which best match the $TE_{10}$-mode of the waveguide are $E_y, H_\rho, H_\psi$, or

$$E_\rho = E_\psi = H_y = 0 \quad \text{and} \quad \frac{\partial}{\partial y} = 0 \quad (7)$$

(Continued)

13-9 (Cont'd)

Using (7) reduces (3) and (4) to

$$-j\omega\mu H_\rho = \frac{1}{\rho}\frac{\partial E_y}{\partial \psi} \tag{8}$$

$$j\omega\mu H_\psi = \frac{\partial E_y}{\partial \rho} \tag{9}$$

Substituting (8) and (9) into (3) leads to

$$-\omega^2\mu\epsilon E_y = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_y}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 E_y}{\partial\psi^2} \quad \text{or} \tag{10}$$

$$\rho^2\frac{\partial^2 E_y}{\partial\rho^2} + \rho\frac{\partial E_y}{\partial\rho} + \frac{\partial^2 E_y}{\partial\psi^2} + (\rho k)^2 E_y = 0 \tag{11}$$

where $k^2 = \omega^2\mu\epsilon$ (11a)

Assuming (11) is separable, we can write

$$E_y(\rho,\psi) = R(\rho)\Psi(\psi) \tag{12}$$

and reduce (11) to

$$\rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + [(k\rho)^2 - p^2]R = 0 \tag{13}$$

$$\frac{\partial^2\Psi}{\partial\psi^2} + p^2\Psi = 0 \tag{14}$$

where $p^2$ is a positive constant.

For the horn, the desired solution of (13) and (14) are of the form [Equation (13) is Bessel's equation, see (V-1)]

$$R(\rho) = A H_p^{(2)}(k\rho) + B H_p^{(1)}(k\rho) \tag{15}$$

$$\Psi(\psi) = C\cos(p\psi) + D\sin(p\psi) \tag{15a}$$

where A, B, C, D are constants. Thus (12) can be written as

$$E_y(\rho,\psi) = [A H_p^{(2)}(k\rho) + B H_p^{(1)}(k\rho)][C\cos(p\psi) + D\sin(p\psi)] \tag{16}$$

Referring to Figure (13.10(b)) and applying the boundary conditions of

$$E_y(\rho, \psi=\psi_h) = E_y(\rho, \psi=-\psi_h) = 0 \tag{17}$$

we find that (18)

$$D=0\,; \quad p = m\left(\frac{\pi}{2\psi_h}\right), \quad m = 1, 3, 5, \cdots$$

and write (16) as

$$E_y(\rho,\psi) = B_m\cos(p\psi)\cdot[H_p^{(2)}(k\rho) + \beta_m H_p^{(1)}(k\rho)] \tag{19}$$

(Continued)

13-9 (Cont'd)

where $B_m$ and $\beta_m$ are constants for the modes $m=1,3,5\cdots$.

The nonvanishing magnetic field components $H_\rho$ and $H_\psi$ of (8) and (9) reduce to

$$H_\rho(\rho,\psi) = -\frac{1}{j\omega\mu}\frac{1}{\rho}\frac{\partial E_y}{\partial \psi} = -jB_m\frac{p}{\omega\mu}\frac{1}{\rho}\sin(p\psi)\left[H_p^{(2)}(k\rho)+\beta_m H_p^{(1)}(k\rho)\right] \quad (20)$$

$$H_\psi(\rho,\psi) = \frac{1}{j\omega\mu}\frac{\partial E_y}{\partial \rho} = -jB_m\frac{k}{\omega\mu}\cos(p\psi)\left[H_p^{(2)\prime}(k\rho)+\beta_m H_p^{(1)\prime}(k\rho)\right],\ ' = \frac{\partial}{\partial(k\rho)} \quad (21)$$

The cylindrical components $H_\rho$ and $H_\psi$ can be resolved to their rectangular counter parts using (VII-7a) or (VII-7b). Thus

$$H_z = H_\rho\cos\psi - H_\psi\sin\psi \quad (22a)$$

$$H_x = H_\rho\sin\psi + H_\psi\cos\psi \quad (22b)$$

which for small flare horns ($\psi$ small) reduce to

$$H_z(\rho,\psi)\simeq H_\rho(\rho,\psi),\quad H_x(\rho,\psi)\simeq H_\psi(\rho,\psi) \quad (23)$$

Assuming the lowest order mode ($m=1$) and only radially outward traveling waves, the fields within the horn can be written as

$$E_\rho = E_\psi = H_y = 0 \quad (24)$$

$$E_y(\rho,\psi) = B_1\cos(p\psi)H_p^{(2)}(k\rho) \quad (25)$$

$$H_x(\rho,\psi) = -jB_1\eta\cos(p\psi)H_p^{(2)\prime}(k\rho) \quad (26)$$

$$H_z(\rho,\psi) = -jB_1\frac{p}{\omega\mu}\frac{1}{\rho}\sin(p\psi)H_p^{(2)}(k\rho) \quad (27)$$

Using the asymptotic form of the Hankel function as given by (V-17), we can write (25) as

$$E_y(\rho,\psi)\simeq j^p B_1\sqrt{\frac{2}{\pi k\rho}}\cos(p\psi)e^{-jk\rho} \quad (28)$$

where $\rho = (z^2+x^2)^{1/2}$ (28a)

using a new coordinate system $(x',y',z')$, as shown in Figure 13.10(a) such that

$$x' = -x,\ y' = -y,\ z' = z-\rho_2,\ \rho_2 = \rho_h\cos\psi_h \quad (29)$$

We can write (28a) as

$$\rho = [z^2+x^2]^{1/2} = [(z'+\rho_2)^2+x'^2]^{1/2} \quad (29a)$$

Equation (29a) can be written, using the binomial expansion, at the aperture of the horn ($z'=0$) as

(Continued)

13-9. (Cont'd)

$$\rho = [\rho_2^2 + x'^2]^{1/2} = \rho_2\left[1 + \left(\frac{x'}{\rho_2}\right)^2\right]^{1/2} = \rho_2\left[1 + \frac{1}{2}\left(\frac{x'}{\rho_2}\right)^2 - \frac{1}{8}\left(\frac{x'}{\rho_2}\right)^4 + \cdots\right] \quad (30)$$

For a small flare angle horns ($x' \ll \rho_2$), (30) can be approximated by

$$\rho \simeq \begin{cases} \rho_2\left[1 + \frac{1}{2}\left(\frac{x'}{\rho_2}\right)^2\right] & \text{for phase terms} \\ \rho_2 & \text{for amplitude terms} \end{cases} \quad (31)$$

Using (31) we can write (28) as

$$E_y'(x', \psi) \simeq E_2 \cos\left(\frac{\pi}{2\psi_h}\right) e^{-j\frac{k}{2}\left(\frac{x'^2}{\rho_2}\right)} \quad (32)$$

where $E_2 = j^P B_1 \sqrt{\frac{2}{\pi k \rho_2}}\, e^{-jk\rho_2}$ (33)

The fields at the aperture of horn can be approximated by

$$E_y'(x') = E_2 \cos\left(\frac{\pi}{a_1}x'\right) e^{-j\frac{k}{2}\left(\frac{x'^2}{\rho_2}\right)} \quad (34)$$

$$H_x'(x') = -\frac{E_2}{\eta}\cos\left(\frac{\pi}{a_1}x'\right) e^{-j\frac{k}{2}\left(\frac{x'^2}{\rho_2}\right)} \quad (35)$$

13-10. $\rho_2 = 6\lambda$, $a_1 = 6\lambda$, $b = 0.25\lambda \Rightarrow \rho_h = \sqrt{\rho_1^2 + (a_1/2)^2} = \sqrt{36+9}\cdot\lambda = 6.7082\lambda$

a. $$D_H = \frac{4\pi b \rho_2}{a_1 \lambda}\left\{[C(u) - C(v)]^2 + [S(u) - S(v)]^2\right\} \quad (13\text{-}41)$$

$$u = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\lambda\rho_2}}{a_1} + \frac{a_1}{\sqrt{\lambda\rho_2}}\right) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{6}}{6} + \frac{6}{\sqrt{6}}\right) = 2.02$$

$$v = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\lambda\rho_2}}{a} - \frac{a_1}{\sqrt{\lambda\rho_2}}\right) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{6}}{6} - \frac{6}{\sqrt{6}}\right) = -1.44$$

Using the tables of Appendix IV

$C(u) = C(2.02) = 0.5069$, $S(u) = S(2.02) = 0.3496$

$C(v) = C(-1.44) = -C(1.44) = -0.50396$

$S(v) = S(-1.44) = -S(1.44) = -0.70712$

$$D_H = \frac{4\pi(0.5)6}{6}\left\{(0.5069 + 0.50396)^2 + (0.3496 + 0.70712)^2\right\} = 6.718 = 8.27\,dB$$

$$A = \frac{a_1}{\lambda}\sqrt{\frac{50}{\rho_h/\lambda}} = 6\sqrt{\frac{50}{6.7082}} = 6(2.7301) = 16.38 \Rightarrow G_H = 84.13 \text{ (from Fig 13.17)}$$

Using (13-42c) $\Rightarrow D_H = \frac{b}{\lambda}\frac{G_H}{\sqrt{\frac{50}{\rho_h/\lambda}}} = 0.25\left(\frac{84.13}{\sqrt{50/6.7082}}\right) = 7.704 = 8.867\,dB$

b. Using (13-33h) $\Rightarrow t = \frac{a_1^2}{8\lambda\rho_2} = \frac{6^2}{8(6)} = \frac{6}{8} = 3/4$

$\theta = 30°$: $\frac{a_1}{\lambda}\sin\theta_1 = 6\cdot\sin(30°) = 3$

Using Figure 13.14 $\Rightarrow E = -14.25 + 20\log_{10}\left(\frac{1+\cos 30°}{2}\right) = -14.25 - 0.6022 = -14.8522$

$\theta = 45°$: $\frac{a_1}{\lambda}\sin\theta_1 = 6\sin(45°) = 4.24 \Rightarrow E = -24 + 20\log_{10}\left(\frac{1+\cos 45°}{2}\right) = -25.3754$

$\theta = 90°$: $\frac{a_1}{\lambda}\sin\theta_1 = 6\sin(90°) = 6 \Rightarrow E = -33.5 + 20\log_{10}\left(\frac{1+\cos 90°}{2}\right) = -39.52$

13-11 $a_1 = \sqrt{3\lambda\rho_2}$

| | Using (13-41C) | Using Figure 13.16 |
|---|---|---|
| $\rho_2 = 6\lambda$ | $\Rightarrow a_1 = 4.2426\lambda$ | $a_1 \simeq 4.25\lambda$ |
| $\rho_2 = 8\lambda$ | $\Rightarrow a_1 = 4.899\lambda$ | $a_1 \simeq 5.0\lambda$ |
| $\rho_2 = 10\lambda$ | $\Rightarrow a_1 = 5.477\lambda$ | $a_1 \simeq 5.5\lambda$ |
| $\rho_2 = 15\lambda$ | $\Rightarrow a_1 = 6.71\lambda$ | $a_1 \simeq 6.9\lambda$ |
| $\rho_2 = 20\lambda$ | $\Rightarrow a_1 = 7.746\lambda$ | $a_1 \simeq 7.75\lambda$ |
| $\rho_2 = 30\lambda$ | $\Rightarrow a_1 = 9.487\lambda$ | $a_1 \simeq 9.5\lambda$ |
| $\rho_2 = 50\lambda$ | $\Rightarrow a_1 = 12.247\lambda$ | $a_1 \simeq 12.4\lambda$ |
| $\rho_2 = 75\lambda$ | $\Rightarrow a_1 = 15.00\lambda$ | $a_1 \simeq 15.25\lambda$ |
| $\rho_2 = 100\lambda$ | $\Rightarrow a_1 = 17.32\lambda$ | $a_1 \simeq 17.6\lambda$ |

13-12. $a = 0.9'' = 2.286\,cm$, $b = 0.4'' = 1.016\,cm$

$f = 11\,GHz \Rightarrow \lambda = \frac{30\times10^9}{11\times10^9} = 2.7273\,cm$

$\frac{a}{\lambda} = \frac{2.286}{2.7273} = 0.8382$, $\frac{b}{\lambda} = \frac{1.016}{2.7273} = 0.37253$

$D_H = 16.3 = 12.12\,dB \Rightarrow \frac{D_E\lambda}{b} = \frac{16.3\lambda}{0.37235\lambda} = 43.755$

Using Figure 13.16 $\Rightarrow \rho_2 \simeq 10\lambda$, $a_1 = 5.5\lambda$

13-13.

a. $a_1 \simeq \sqrt{3\lambda\rho_2} = \sqrt{3\lambda\cdot(10\lambda)} = \sqrt{30\lambda^2} = 5.4772\lambda$

$\rho_h = \sqrt{\rho_2^2 + (a_1/2)^2} = \sqrt{(10)^2 + (5.4772/2)^2}\,\lambda = 10.3682\lambda$

$\psi_h = \tan^{-1}\left(\frac{a_1/2}{\rho_2}\right) = \tan^{-1}\left(\frac{2.7386}{10}\right) = 15.3156^\circ = 0.2739$ (rads)

b. $A = \frac{a_1}{\lambda}\sqrt{\frac{50}{\rho_h/\lambda}} = 5.4772\sqrt{\frac{50}{10.3682}} = 5.4772\,(2.1960) = 12.0280$

$A = 12.0280 \Rightarrow$ From Figure 13.17 $\Rightarrow G_H \simeq 98.8$

$D_H = \frac{b}{\lambda}\frac{G_H}{\sqrt{\frac{50}{\rho_h/\lambda}}} = 0.3387\,\frac{98.8}{\sqrt{\frac{50}{10.3682}}} = 0.3387\,\frac{(98.8)}{2.1960} = 15.2384 = 11.8294\,dB$

c. $\epsilon_{ap} = \frac{A_{em}}{A_p} = \frac{1.2126\lambda^2}{0.3387\lambda(5.4772\lambda)} = 0.6537 = 65.37\%$

$A_{em} = \frac{\lambda^2}{4\pi}G_0 = \frac{\lambda^2}{4\pi}D_0 = \frac{\lambda^2}{4\pi}(15.2384) = 1.2126\lambda^2$

d. $\Delta\phi_{max} = k(\rho_h - \rho_2) = \frac{2\pi}{\lambda}(10.3682-10)\lambda = 2\pi\,(0.3682)$

$= 2.3135$ rads $= 132.56^\circ$

13-14. Referring to Figure 13.18

$$\rho_e^2 = \rho_1^2 + \left(\frac{b_1}{2}\right)^2 \Rightarrow \rho_1^2 = \rho_e^2 - \left(\frac{b_1}{2}\right)^2 \Rightarrow \rho_1 = \left[\rho_e^2 - \left(\frac{b_1}{2}\right)^2\right]^{1/2} = b_1\left[\left(\frac{\rho_e}{b_1}\right)^2 - \frac{1}{4}\right]^{1/2}$$

Also

$$\rho_1 = P_e + \frac{b}{2}\cot(\psi_e) = P_e + \frac{b}{2}\frac{\rho_1}{(b_1/2)} = P_e + \rho_1\left(\frac{b}{b_1}\right) \Rightarrow P_e = \rho_1\left(1 - \frac{b}{b_1}\right)$$

Thus

$$P_e = \rho_1\left(1 - \frac{b}{b_1}\right) = b_1\left[\left(\frac{\rho_e}{b_1}\right)^2 - \left(\frac{1}{4}\right)\right]^{1/2}\left(1 - \frac{b}{b_1}\right) = (b_1 - b)\sqrt{\left(\frac{\rho_e}{b_1}\right)^2 - \frac{1}{4}}$$

In a similar manner, we can show that

$$P_h = (a_1 - a)\sqrt{\left(\frac{\rho_h}{a_1}\right)^2 - \frac{1}{4}}$$

13-15

$\rho_1 = 13.5'' = 34.49$ cm
$\rho_2 = 14.2'' = 36.07$ cm
$a_1 = 7.65'' = 19.43$ cm
$b_1 = 5.65'' = 14.35$ cm

$\Rightarrow$ $\rho_e = [\rho_1^2 + (b_1/2)^2]^{1/2} = 13.7924'' = 35.0327$ cm
$\rho_h = [\rho_1^2 + (a_1/2)^2]^{1/2} = 14.7061'' = 37.3536$ cm

$a = 0.9'' = 2.286$ cm
$b = 0.4'' = 1.016$ cm

a. $P_e = (b_1 - b)\sqrt{\left(\frac{\rho_e}{b_1}\right)^2 - \frac{1}{4}} = (5.65 - 0.4)\sqrt{\left(\frac{13.7924}{5.65}\right)^2 - \frac{1}{4}} = 12.544'' = 31.862$ cm

$P_h = (a_1 - a)\sqrt{\left(\frac{\rho_h}{a_1}\right)^2 - \frac{1}{4}} = (7.65 - 0.9)\sqrt{\left(\frac{14.7061}{7.65}\right)^2 - \frac{1}{4}} = 12.529'' = 31.8246$ cm

Therefore $P_e \simeq P_h$, and the pyramidal horn is physically realizable

b. $f = 8.2$ GHz $\Rightarrow \lambda = \frac{30 \times 10^9}{8.2 \times 10^9} = 3.658537$ cm $= 1.44''$

$\rho_1 = \frac{13.5}{1.44}\lambda = 9.375\lambda$, $a_1 = \frac{7.65}{1.44}\lambda = 5.3125\lambda$, $a = \frac{0.9}{1.44}\lambda = 0.625\lambda$

$\rho_e = \frac{13.7924}{1.44}\lambda = 9.578''$

$\rho_2 = \frac{14.2}{1.44}\lambda = 9.861\lambda$, $b_1 = \frac{5.65}{1.44}\lambda = 3.9236\lambda$, $b = \frac{0.4}{1.44}\lambda = 0.278\lambda$

$\rho_h = \frac{14.7061}{1.44}\lambda = 10.213\lambda$

$$u = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\lambda\rho_2}}{a_1} + \frac{a_1}{\sqrt{\lambda\rho_2}}\right) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{9.861}}{5.3125} + \frac{5.3125}{\sqrt{9.861}}\right) = 1.6142$$

$$v = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\lambda\rho_2}}{a_1} - \frac{a_1}{\sqrt{\lambda\rho_2}}\right) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{9.861}}{5.3125} - \frac{5.3125}{\sqrt{9.861}}\right) = -0.7783$$

$$\frac{b_1}{\sqrt{2\lambda\rho_2}} = \frac{3.9236}{\sqrt{2 \cdot (9.861)}} = 0.8835$$

$C(u) = C(1.6142) = 0.3595$
$S(u) = S(1.6142) = 0.6262$
$C(v) = C(-0.7783) = -C(0.7783) = -0.7091$
$S(v) = S(-0.7783) = -S(0.7783) = -0.2326$

} From Table in Appendix IV

(Continued)

13-15 (Cont'd)

$$C\left(\frac{b_1}{\sqrt{2\lambda\rho_2}}\right) = C(0.8835) = 0.7579$$
$$S\left(\frac{b_1}{\sqrt{2\lambda\rho_2}}\right) = S(0.8835) = 0.3249$$
} From Table in Appendix IV

Using (13-52) or (13-52a)

$$D_p = \frac{8\pi\rho_1\rho_2}{a_1 \cdot b_1}\left\{[C(u)-C(v)]^2+[S(u)-S(v)]^2\right\}\left\{C^2\left(\frac{b_1}{\sqrt{2\lambda\rho_1}}\right)+S^2\left(\frac{b_1}{\sqrt{2\lambda\rho_1}}\right)\right\}$$
$$= \frac{8\pi(13.5)(14.2)}{7.65\,(5.65)}\left[(0.3595+0.7091)^2+(0.6262+0.2326)^2\right]\left[(0.7579)^2+(0.3249)^2\right]$$

$D_p = 142.456 = 21.54\,dB$

Using (13-53) $\Rightarrow D_p = 10\left[1.008+\log_{10}\left(\frac{a_1 b_1}{\lambda^2}\right)\right]-(L_e+L_h)$

$$S = \frac{b_1^2}{8\lambda\rho_1} = \frac{(3.9236)^2}{8(9.375)} = 0.205$$
$$t = \frac{a_1^2}{8\lambda\rho_2} = \frac{(5.3125)^2}{8(9.861)} = 0.358$$
} Using Figure 13.23 $\Rightarrow L_e = 0.5\,dB$, $L_h = 0.8\,dB$

Thus $D_p = 10\{1.008+\log_{10}[5.3125(3.9236)]-(0.5+0.8) = 23.2698-1.3 = 21.96\,dB$

$$A = \frac{a_1}{\lambda}\sqrt{\frac{50}{\rho_h/\lambda}} = 5.3125\sqrt{\frac{50}{10.213}} = 11.7546 \Rightarrow G_H = 98.254 \text{ from Figure 13.17}$$

$$B = \frac{b_1}{\lambda}\sqrt{\frac{50}{\rho_e/\lambda}} = 3.9236\sqrt{\frac{50}{9.5778}} = 8.965 \Rightarrow G_E = 85.325 \text{ from Figure 13.9}$$

Using (13-54e) $D_p = \frac{G_E G_H}{10.1859\sqrt{\frac{50}{\rho_e/\lambda}}\sqrt{\frac{50}{\rho_h/\lambda}}} = \frac{(85.325)(98.254)}{(10.1859)(2.285)(2.2126)} = 162.794$

$D_p = 162.794 = 22.17\,dB$

$f = 10.3\,GHz: \Rightarrow \lambda = \frac{30\times10^9}{10.3\times10^9} = 2.9126\,cm = 1.1467''$

Using the same procedure as for $f = 8.2\,GHz$, we have that

$\rho_1 = 11.1773\lambda$, $\rho_2 = 12.383\lambda$, $\rho_e = 12.0279\lambda$, $\rho_h = 12.8247\lambda$

$a_1 = 6.671\lambda$, $b_1 = 4.927\lambda$, $a = 0.7849\lambda$, $b = 0.3488\lambda$

$\mu = 1.7135$, $v = -0.9675$, $\frac{b_1}{\sqrt{2\lambda\rho_1}} = W = 1.015$

$C(u) = 0.32515$, $C(v) = -0.7750$, $C(w) = 0.780$

$S(u) = 0.5359$, $S(v) = -0.4063$, $S(w) = 0.440$

Using (13-52) $\Rightarrow D_p = \frac{8\pi(13.5)14.2}{(7.65)(5.65)}\left\{(0.32515+0.7750)^2+(0.5359+0.4063)^2\right\}\times$

$$\cdot\left\{(0.78)^2+(0.44)^2\right\} = 187.56 = 22.73\,dB$$

(Continued)

13-15 (Cont'd)

$$S=\frac{b_1^2}{8\lambda\rho_1}=0.2577\Rightarrow L_e=0.9\ ;\ t=\frac{a_1^2}{8\lambda\rho_2}=0.4492\Rightarrow L_h=1.5$$

Using (13-53) $\Rightarrow D_p=10\{1.008+\log_{10}[6.671\,(4.927)]\}-(0.9+1.5)=22.847\text{dB}$

$A=6.671\,(1.97452)=13.172\Rightarrow G_H=98.673$

$B=4.927\,(2.03887)=10.0455\Rightarrow G_E=81.550$

Using (13-54e) $\Rightarrow D_p=\dfrac{98.673\,(81.550)}{10.1859\,(1.97452)\,2.03887}=196.2329=22.93\text{ dB}$

$f=12.4\text{ GHz}:\ \lambda=\dfrac{30\times10^9}{12.4\times10^9}=2.41935\text{ cm}=0.9525''$

Using the same procedure as for $f=8.2$ GHz, we have that

$\rho_1=14.1173\lambda,\ \rho_2=14.908\lambda,\ \rho_e=14.48\lambda,\ \rho_h=15.4395\lambda$

$a_1=8.0315\lambda,\ b_1=5.9318\lambda,\ a=0.9449\lambda,\ b=0.4199\lambda$

$u=1.811,\ v=-1.131,\ w=\dfrac{b_1}{\sqrt{2\lambda\rho_1}}=1.114$

$C(u)=0.340,\ C(v)=-0.745,\ C(w)=0.760$

$S(u)=0.443,\ S(v)=-0.560,\ S(w)=0.545$

Using (13-52) $\Rightarrow D_p=\dfrac{8\pi(13.5)(14.2)}{7.65\cdot(5.65)}\left\{(0.340+0.745)^2+(0.443+0.560)^2\right\}\times$

$$[(0.76)^2+(0.545)^2]=212.84=23.28\text{dB}$$

$$S=\frac{b_1^2}{8\lambda\rho_1}=0.31\Rightarrow L_e=1.4\ ;\ t=\frac{a_1^2}{8\lambda\rho_2}=0.5411\Rightarrow L_h=2.1$$

Using (13-53) $\Rightarrow D_p=10\{1.008+\log_{10}[8.0315\,(5.9318)]\}-(1.4+2.1)=23.36\text{dB}$

$A=8.0318\,(1.79956)=14.453\Rightarrow G_H=95.1$

$B=5.9318\,(1.858)=11.0227\Rightarrow G_E=80.6$

Using (13-54e) $\Rightarrow D_p=\dfrac{95.1\,(80.6)}{(10.1859)(1.79956)(1.858)}=225.063=23.52\text{ dB}$

In summary, all three equations yield nearly identical results.

The computed directivities agree closely with those of a commercial unit.

→ Computer Result

At 8.2 GHz → $D_p=21.7017\text{dB}$

10.3 GHz → $D=22.7914\text{dB}$

12.4 GHz → $D_p=23.3311\text{ dB}$

13-16. $\rho_1 = 5.3'' = 13.46\ cm$
$\rho_2 = 6.2'' = 15.748\ cm$
$a_1 = 3.09'' = 7.85\ cm$
$b_1 = 2.34'' = 5.944\ cm$
$a = 0.9'' = 2.286\ cm$
$b = 0.4'' = 1.016\ cm$

$\rho_e = [\rho_1^2 + (b_1/2)^2]^{1/2} = 5.428'' = 13.786\ cm$

$\rho_h = [\rho_1^2 + (a_1/2)^2]^{1/2} = 6.3896'' = 16.2296\ cm$

a. $P_e = (b_1 - b)\sqrt{(\frac{\rho_e}{b_1})^2 - \frac{1}{4}} = (2.34 - 0.4)\sqrt{(\frac{5.428}{2.34})^2 - \frac{1}{4}} = 4.394'' = 11.16\ cm$

$P_h = (a_1 - a)\sqrt{(\frac{\rho_h}{a_1})^2 - \frac{1}{4}} = (3.09 - 0.9)\sqrt{(\frac{6.3896}{3.09})^2 - \frac{1}{4}} = 4.394'' = 11.16\ cm$

Therefore $P_e = P_h$, and the pyramidal horn is physically realizable.

b. $f = 8.2\ GHz \Rightarrow \lambda = \frac{30 \times 10^9}{8.2 \times 10^9} = 3.658537\ cm = 1.44''$

$\rho_1 = \frac{5.3}{1.44}\lambda = 3.68\lambda$, $a_1 = \frac{3.09}{1.44}\lambda = 2.1458\lambda$, $a = \frac{0.9}{1.44}\lambda = 0.625\lambda$,

$\rho_e = \frac{5.428}{1.44}\lambda = 3.764\lambda$

$\rho_2 = \frac{6.2}{1.44}\lambda = 4.305\lambda$, $b_1 = \frac{2.34}{1.44}\lambda = 1.625\lambda$, $b = \frac{0.4}{1.44}\lambda = 0.278\lambda$,

$\rho_h = \frac{6.3896}{1.44} = 4.437\lambda$

$u = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\lambda\rho_2}}{a_1} + \frac{a_1}{\sqrt{\lambda\rho_2}}\right) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{4.305}}{2.1458} + \frac{2.1458}{\sqrt{4.305}}\right) = 1.415$

$v = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\lambda\rho_2}}{a_1} - \frac{a_1}{\sqrt{\lambda\rho_2}}\right) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{4.305}}{2.1458} - \frac{2.1458}{\sqrt{4.305}}\right) = -0.0476$

$\frac{b_1}{\sqrt{2\lambda\rho_2}} = \frac{1.625}{\sqrt{2(3.68)}} = 0.5989$

$C(u) = C(1.415) = 0.5284$
$S(u) = S(1.415) = 0.7111$
$C(v) = C(-0.0476) = -C(0.0476) = -0.0476$
$S(v) = S(-0.0476) = -S(0.0476) = -0.00025$

} From Tables in Appendix IV

$C(0.5989) = 0.5801$, $S(0.5989) = 0.11004$

Using (13-52) or (13-52a)

$$D_p = \frac{8\pi\rho_1\rho_2}{a_1 b_1}\left\{[C(u) - C(v)]^2 + [S(u) - S(v)]^2\right\}\left\{C^2\left(\frac{b_1}{\sqrt{2\lambda\rho_1}}\right) + S^2\left(\frac{b_1}{\sqrt{2\lambda\rho_1}}\right)\right\}$$

$$= \frac{8\pi(5.3)(6.2)}{(3.09)(2.34)}\left[(0.5284 + 0.0476)^2 + (0.7111 + 0.00025)^2\right]\left[(0.5801)^2 + (0.11004)^2\right]$$

$$= 33.36 = 15.23\ dB$$

(Continued)

13-16 (Cont'd)

Using (13-53) $\Rightarrow D_p = 10\left[1.008 + \log_{10}\left(\frac{a_1 b_1}{\lambda^2}\right)\right] - (L_e + L_h)$

$S = \frac{b_1^2}{8\lambda\rho_1} = \frac{(1.624)^2}{8(3.679)} = 0.0896$

$t = \frac{a_1^2}{8\lambda\rho_2} = \frac{(2.1457)^2}{8(4.304)} = 0.1337$

Using Figure 13.23 $\Rightarrow L_e = 0.1\,dB$, $\Rightarrow L_h = 0.2\,dB$

Thus $D_p = 10\{1.008 + \log_{10}[2.1458(1.625)]\} - (0.1+0.2) = 15.5 - 0.3 = 15.2\,dB$

$A = \frac{a_1}{\lambda}\sqrt{\frac{50}{\rho_h/h}} = 2.1458\sqrt{\frac{50}{4.437}} = 7.2037 \Rightarrow G_h = 71.29$ from Figure 13.17

$B = \frac{b_1}{\lambda}\sqrt{\frac{50}{\rho_e/h}} = 1.625\sqrt{\frac{50}{3.769}} = 5.9158 \Rightarrow G_E = 58.66$ from Figure 13.9

Using (13-54e) (Computer Result $\Rightarrow$ 15.2507dB)

$$D_p = \frac{G_E \cdot G_H}{10.1859\sqrt{\frac{50}{\rho_e/\lambda}}\sqrt{\frac{50}{\rho_h/\lambda}}} = \frac{58.66(71.29)}{10.1859(3.64275)(3.3573)} = 33.57 = 15.26\,dB$$

$f = 10.3\,GHz : \lambda = \frac{30\times10^9}{10.3\times10^9} = 2.9126\,cm = 1.1467''$

Using the same procedure as for $f = 8.2\,GHz$, we have that

$\rho_1 = 4.622\lambda$, $\rho_2 = 5.4068\lambda$, $\rho_e = 4.7336\lambda$, $\rho_h = 5.5722\lambda$

$a_1 = 2.695\lambda$, $b_1 = 2.041\lambda$, $a = 0.7849\lambda$, $b = 0.3488\lambda$

$u = 1.4296$, $v = -0.2095$, $W = \frac{b_1}{\sqrt{2\lambda\rho_1}} = 0.6713$

$C(u) = 0.510$, $C(v) = -0.209$, $C(W) = 0.6371$

$S(u) = 0.709$, $S(v) = -0.005$, $S(W) = 0.1545$

Using (13-52) $\Rightarrow D_p = \frac{8\pi(5.3)6.2}{3.09(2.34)}\left\{(0.510+0.209)^2 + (0.709+0.005)^2\right\} \times \left\{(0.6371)^2 + (0.1545)^2\right\} = 50.4 = 17.02\,dB$

$D_p = 17.02\,dB$

$S = \frac{b_1^2}{8\lambda\rho_1} = 0.1126 \Rightarrow L_e = 0.2$ : $t = \frac{a_1^2}{8\lambda\rho_2} = 0.1679 \Rightarrow L_h = 0.25$

Using (13-53) $\Rightarrow D_p = 10\{1.008 + \log_{10}[2.695(2.041)]\} - (0.2+0.25) = 17.034\,dB$

$A = 2.695(2.9955) = 8.0729 \Rightarrow G_H = 78.6$

$B = 2.041(3.25) = 6.633 \Rightarrow G_E = 66.1$

(Computer Program 17.0838dB)

Using (13-54e) $\Rightarrow D_p = \frac{66.1(78.6)}{10.1859(2.9955)3.25} = 52.39 = 17.19\,dB$

(Continued)

13-16. (Cont'd)

$f = 12.4\,GHz \Rightarrow \lambda = \frac{30\times10^9}{12.4\times10^9} = 2.41935\,cm = 0.9525''$

Using the same procedure as for $f = 8.2\,GHz$, we have that

$\rho_1 = 5.5643\lambda$, $\rho_2 = 6.5092\lambda$, $\rho_e = 5.6987\lambda$, $\rho_h = 6.7082\lambda$

$a_1 = 3.2441\lambda$, $b_1 = 2.4567\lambda$, $a = 0.9449\lambda$, $b = 0.4199\lambda$

$u = 1.455$, $v = -0.343$, $W = \frac{b_1}{\sqrt{2\lambda\rho_1}} = 0.7364$

$C(u) = 0.490$, $C(v) = -0.339$, $C(W) = 0.685$

$S(u) = 0.705$, $S(v) = -0.0225$, $S(W) = 0.200$

Using (13-52) $\Rightarrow D_p = \frac{8\pi(5.3)6.2}{3.09(2.34)}\left\{(0.99+0.339)^2+(0.705+0.0225)^2\right\}\left[(0.685)^2+(0.2)^2\right]$

$D_p = 70.75 = 18.497\,dB$ [Computer Result $= 18.5150\,dB$]

$s = \frac{b_1^2}{8\lambda\rho_1} = 0.136 \Rightarrow Le = 0.25$; $t = \frac{a_1^2}{8\lambda\rho_2} = 0.202 \Rightarrow Lh = 0.3$

Using (13-53) $\Rightarrow D_p = 10\{1.008 + \log_{10}[3.2441(2.4567)]\} - (0.25+0.3) = 18.544\,dB$

$A = 3.2441(2.7301) = 8.8568 \Rightarrow G_H = 84.55$

$B = 2.4567(2.9621) = 7.2769 \Rightarrow G_E = 69.65$

Using (13-54e) $\Rightarrow D_p = \frac{69.65(84.55)}{10.1859(2.7301)(2.9621)} = 71.492 = 18.54\,dB$

In summary, all three equations [i.e. (13-52), (13-53), and (13-54e)] yield nearly identical results. The computed directivities agree closely with those measured of the commercially available unit of Fig 13-24.

13-17. $G_o = 17.05\,dB = 10\log_{10}G_o\,(dimensionless) \Rightarrow G_o = 10^{1.705} = 50.7$

$f = 11\,GHz. \Rightarrow \lambda = \frac{30\times10^9}{11\times10^9} = 2.7273\,cm$

$a = \frac{2.286\lambda}{2.7273} = 0.8382\lambda$, $b = \frac{1.016}{2.7273}\lambda = 0.3725\lambda$

The initial value of $\chi$, is taken, using (13-57), as

$\chi_1 = \frac{50.7}{2\pi\sqrt{2\pi}} = 3.219127$

which does not satisfy (13-56). After few iteration, it is found that $\chi_1 = 2.96795$ is a more accurate value. Thus

$\rho_e = \chi\lambda = 2.96795(2.7273) = 8.0945\,cm = 3.1868'' = 2.96795\lambda$

$\rho_h = \frac{G_o^2}{8\pi^3}\left(\frac{1}{\chi}\right)\lambda = \frac{(50.7)^2}{8\pi^3}\left(\frac{1}{2.96795}\right)\lambda = 3.49156\lambda = 9.5225\,cm = 3.749''$

(Continued)

13-17 (Cont'd)

$$a_1 = \frac{G_0}{2\pi}\sqrt{\frac{3}{2\pi\chi}}\,\lambda = \frac{50.7}{2\pi}\sqrt{\frac{3}{2\pi(2.96795)}}\,\lambda = 3.23646\lambda = 8.8268\text{ cm} = 3.475''$$

$$b_1 = \sqrt{2\chi}\,\lambda = \sqrt{2(2.96795)}\,\lambda = 2.43637\lambda = 6.6447\text{ cm} = 2.616''$$

$$\rho_e = (b_1 - b)\left[\left(\frac{\rho_e}{b_1}\right)^2 - \frac{1}{4}\right]^{1/2} = 6.25263\text{ cm} = 2.46167''$$

$$\rho_h = (a_1 - a)\left[\left(\frac{\rho_h}{a_1}\right)^2 - \frac{1}{4}\right]^{1/2} = 6.25269\text{ cm} = 2.46169''$$

$$\Rightarrow \rho_e \simeq \rho_h \simeq 6.2526\text{ cm} \simeq 2.4617''$$

Using (13-54a) $\Rightarrow A = \frac{a_1}{\lambda}\sqrt{\frac{50}{\rho_h/\lambda}} = 12.247 \Rightarrow G_{TH} = 98.921$ from Figure 13.17

Using (13-54b) $\Rightarrow B = \frac{b_1}{\lambda}\sqrt{\frac{50}{\rho_e/\lambda}} = 9.9999 \Rightarrow G_{TE} = 81.518$ from Figure 13.9

Using (13-54e)

$$D_p = \frac{G_{TE}\cdot G_{TH}}{10.1859\sqrt{\frac{50}{\rho_e/\lambda}}\sqrt{\frac{50}{\rho_h/\lambda}}} = \frac{81.518\,(98.921)}{10.1859\sqrt{\frac{50}{2.96795}}\sqrt{\frac{50}{3.49156}}} = 50.9695 = 17.07\text{dB}$$

13-18. $\lambda = \frac{30\times10^9}{10\times10^9} = 3\text{ cm}$

a. $a_1 \simeq \sqrt{3\lambda\rho} = \sqrt{3\lambda(10\lambda)} = \sqrt{30\lambda^2} = 5.477\lambda = 16.43\text{ cm}$

$b_1 \simeq \sqrt{2\lambda\rho} = \sqrt{20\lambda^2} = 4.472\lambda = 13.416\text{ cm}$

b. $G_0 = \frac{1}{2}\frac{4\pi}{\lambda^2}(a_1 b_1) = \frac{1}{2}\frac{4\pi}{\lambda^2}(5.477\lambda)(4.472\lambda) = 153.89 = 21.87\text{dB}$

c. $e_r e_{cd}\,\varepsilon_{ap} = 1\cdot 1\cdot\varepsilon_{ap} = \frac{1}{2}$, $\varepsilon_{ap} = \frac{1}{2} = 50\,\%$

d. $A_{em} = \frac{\lambda^2}{4\pi}G_0 = \frac{3^2}{4\pi}(153.89) = 110.2156\text{ cm}^2 = 110.2156\times10^{-4}\text{ m}^2$

$P_{rec} = W^i A_{em} = 10\times10^{-6}\times110.2156\times10^{-4} = 1,102.156\times10^{-10} = 11.02156\times10^{-8}$

$P_{rec} = 11.02156\times10^{-8} = 0.1102156\ \mu\text{ Watts}$

13-19. The problem was solved using compuler program.

$\chi = 6.047$

$\rho_e = 37.0\text{ cm}$

$\rho_h = 40.8\text{ cm}$

$a_1 = 27.4\text{ cm}$

$b_1 = 21.3\text{ cm}$

$\rho_e = \rho_h = 31.7\text{ cm}$

13-20. $d_m = \sqrt{3\ell\lambda} \Rightarrow \ell = \sqrt{L^2 + (d_m/2)^2}$

| From Figure 13.25 | | | Using $d_m = \sqrt{3\ell\lambda}$ |
|---|---|---|---|
| $L = 75\lambda$ | $\Rightarrow d_m = 14.5\lambda$ | $\Rightarrow \ell = 73.35\lambda$ | $d_m = 15.03\lambda$ |
| $L = 50\lambda$ | $\Rightarrow d_m = 12\lambda$ | $\Rightarrow \ell = 50.36\lambda$ | $d_m = 12.29\lambda$ |
| $L = 30\lambda$ | $\Rightarrow d_m = 9.4\lambda$ | $\Rightarrow \ell = 30.37\lambda$ | $d_m = 9.54\lambda$ |
| $L = 20\lambda$ | $\Rightarrow d_m = 7.6\lambda$ | $\Rightarrow \ell = 20.36\lambda$ | $d_m = 7.81\lambda$ |
| $L = 15\lambda$ | $\Rightarrow d_m = 6.7\lambda$ | $\Rightarrow \ell = 15.37\lambda$ | $d_m = 6.79\lambda$ |
| $L = 10\lambda$ | $\Rightarrow d_m = 5.6\lambda$ | $\Rightarrow \ell = 10.38\lambda$ | $d_m = 5.58\lambda$ |
| $L = 8\lambda$ | $\Rightarrow d_m = 5\lambda$ | $\Rightarrow \ell = 8.38\lambda$ | $d_m = 5.01\lambda$ |
| $L = 6\lambda$ | $\Rightarrow d_m = 4.2\lambda$ | $\Rightarrow \ell = 6.36\lambda$ | $d_m = 4.37\lambda$ |
| $L = 4\lambda$ | $\Rightarrow d_m = 3.5\lambda$ | $\Rightarrow \ell = 4.37\lambda$ | $d_m = 3.62\lambda$ |
| $L = 2\lambda$ | $\Rightarrow d_m = 2.5\lambda$ | $\Rightarrow \ell = 2.36\lambda$ | $d_m = 2.66\lambda$ |
| $L = \lambda$ | $\Rightarrow d_m = 1.9\lambda$ | $\Rightarrow \ell = 1.38\lambda$ | $d_m = 2.03\lambda$ |
| $L = 0.5\lambda$ | $\Rightarrow d_m = 1.5\lambda$ | $\Rightarrow \ell = 0.9\lambda$ | $d_m = 1.64\lambda$ |

13-21. $L = 19.5''$, $d_m = 15''$, $d = 2.875''$

a. From Figure 13.26 $\ell = \sqrt{L^2 + (d_m/2)^2} = 20.89''$

The optimum gain will occur when, according to (13-60)

$$d_m = \sqrt{3\ell\lambda} \Rightarrow d_m^2 = 3\ell\lambda \Rightarrow \lambda = d_m^2/3\ell = (15)^2/[3(20.89)] = 3.5898'' = 9.118\ \text{cm}$$

$$f = \frac{c}{\lambda} = \frac{3\times10^{10}}{9.118} = 3.29\times10^9\ \text{Hz} = 3.29\ \text{GHz}$$

$$L = \frac{19.5''}{3.5898''}\lambda = 5.432\lambda\ ,\quad d_m = \frac{15''}{3.5898''}\lambda = 4.1785\lambda$$

For these two L and $d_m$, from (Fig 13.27). $D_c \simeq 19.5\text{dB}$

According to (13-59b) and (13-59c), $S = \frac{d_m^2}{8\lambda\ell} = \frac{3}{8}$

$$L(S) \cong (0.8 - 1.71S + 26.2S^2 - 17.79S^3) = 2.9\text{dB}$$

Using these, we can compute the directivity from (13-59).

Thus $D_c = 10\log_{10}\left(\frac{\pi d_m}{\lambda}\right)^2 - 2.9 = 10\log_{10}[\pi(4.1785)]^2 - 2.9$

$= 22.363 - 2.9 = 19.46\text{dB}$

which agrees closely with the value obtained from Fig 13-27

(Continued)

13-21. (Cont'd) $f=2.5GHz \rightarrow \lambda = \frac{c}{f} = \frac{30\times10^9}{2.5\times10^9} = 12\,cm = 4.7244''$

From Fig 13-27 — From Fig 13-27

$L = \frac{19.5}{4.7244}\lambda = 4.1275\lambda$, $dm = \frac{15}{4.7244}\lambda = 3.175\lambda$, $D_c \cong 18.5\,dB$

Using equations

$(13-59c) \Rightarrow S = \frac{dm^2}{8\lambda l} = \frac{(15)^2}{8(4.7244)(20.89)} = 0.284975$ (Wavelengths)

$(13-59b) \Rightarrow L(S) = [0.8 - 1.71(S) + 26.25S^2 - 17.79S^3] = 2.033$

$(13-59) \Rightarrow D_c = 10\log_{10}[\pi(3.175)]^2 - 2.033 = 19.978 - 2.033$

$D_c = 17.94dB$

$f = 5GHz \Rightarrow \lambda = \frac{30\times10^9}{5\times10^9} = 6\,cm = 2.3622''$

Using Fig 13.27

$L = \frac{19.5}{2.3622}\lambda = 8.255\lambda$, $dm = \frac{15}{2.3622}\lambda = 6.35\lambda$, $D_c \simeq 20dB$ (From Fig 13.27)

Using equations

$(13-59c) \Rightarrow S = \frac{dm^2}{8\lambda l} = \frac{(15)^2}{8(2.3622)(20.89)} = 0.56995$ (wavelengths)

$(13-59b) \Rightarrow L(S) = [0.8 - 1.71S + 26.25S^2 - 17.79S^3] = 5.0588$

$(13-59) \Rightarrow D_c = 10\log_{10}[\pi(6.35)]^2 - 5.0588 = 25.998 - 5.0588 = 20.94dB$

c. The cutoff frequency of the dominant $TE_{11}$-mode of a circular waveguide is given by (11-62) or

$(f_c)_{11} = \frac{\chi'_{11}}{2\pi a\sqrt{\mu\epsilon}}$ where $\chi'_{11}$ is the first zero of the derivative of the Bessel function of the first kind of order one [i.e. $J_1'(\chi'_{11}) = 0$], and it is equal to $\chi'_{11} = 1.841$

Thus $(f_c)_{11} = \frac{1.841}{2\pi(d/2)\sqrt{\mu\epsilon}} = \frac{1.841}{\pi d\sqrt{\mu\epsilon}} = \frac{1.841\,c}{\pi d}$

For $d = 2.875'' = 7.3025cm$

$(f_c)_{11} = \frac{1.841(30\times10^9)}{\pi(7.3025)} = 2.4074\times10^9 = 2.4074\,GHz$

13-22. $d_m = \sqrt{3\ell\lambda} \Rightarrow S = \frac{d_m^2}{8\lambda\ell} = \frac{3}{8}$ . ( From (13-59b) $\Rightarrow L(s=3/8) \simeq 2.91\,dB$ )

From (13-59) $\Rightarrow D_c = 22.6 = 10\log_{10}\left(\frac{c}{\lambda}\right)^2 - 2.91 \Rightarrow 10\log_{10}\left(\frac{c}{\lambda}\right)^2 = 25.51$

$\log_{10}\left(\frac{c}{\lambda}\right)^2 = 2.551 \Rightarrow \left(\frac{c}{\lambda}\right)^2 = 10^{2.551} = 355.795$

$\frac{c}{\lambda} = \frac{\pi d_m}{\lambda} = \sqrt{355.795} = 18.8625 \Rightarrow d_m = \frac{18.8625\lambda}{\pi}$

$d_m = \frac{18.8625\lambda}{\pi} = 6.00\lambda$

$\lambda = \frac{30 \times 10^9}{11 \times 10^9} = 2.7273\ cm$

$d_m = 6\lambda = 6(2.7273) = 16.3636\ cm = 6.44''$

Since $d_m = \sqrt{3\ell\lambda} \Rightarrow \ell = \frac{d_m^2}{3\lambda} = \frac{(16.3636)^2}{3(2.7273)} = 32.727\ cm = 12.8847''$

$L = \sqrt{\ell^2 - (d_m/2)^2} = \sqrt{(32.727)^2 - (16.3636/2)^2} = 31.6878\ cm = 12.475''$

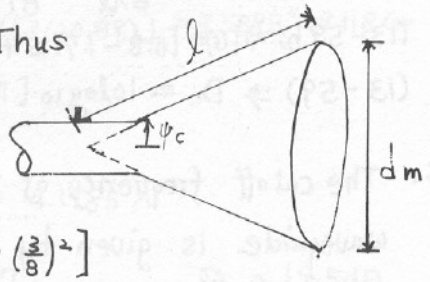
$\psi_c = \tan^{-1}\left(\frac{d_m/2}{L}\right) = \tan^{-1}\left(\frac{16.3636}{2(31.6878)}\right) = 14.475^\circ \Rightarrow 2\psi_c = 28.95^\circ$

13-23. $D_c(dB) = 10\log_{10}\left[\epsilon_{ap}\frac{4\pi}{\lambda^2}(\pi a^2)\right] = 10\log_{10}\left(\frac{c}{\lambda}\right) - L(s)$

$L(s) \simeq (0.8 - 1.71S + 26.25S^2 - 17.79S^3)$

$S = \frac{d_m^2}{8\lambda\ell}$ , $\lambda = \frac{30\times10^9}{10\times10^9} = 3\ cm$

For optimum directivity, $d_m \simeq \sqrt{3\ell\lambda}$. Thus




$S = \frac{3\ell\lambda}{8\ell\lambda} = \frac{3}{8}$

For $S = \frac{3}{8}$

$L(s) \simeq \left[0.8 - 1.71\left(\frac{3}{8}\right) + 26.25\left(\frac{3}{8}\right)^2 - 17.79\left(\frac{3}{8}\right)^2\right]$

$\simeq [0.8 - 0.6412S + 26.25(0.1406) - 17.79(0.0527)]$

$L(s) \simeq 2.912\,dB$

$D_c(dB) = 20 = 10\log_{10}\left(\frac{c}{\lambda}\right)^2 - 2.912$

$22.912 = 10\log_{10}(c/\lambda)^2$ , $(c/\lambda)^2 = 10^{2.2912} = 195.52 = \left(\frac{2\pi a}{\lambda}\right)^2 = \left(\frac{d_m\pi}{\lambda}\right)^2$

$d_m = \frac{\sqrt{195.52}}{\pi}\cdot\lambda = 4.4508\lambda = 4.4508(3) = 13.3526\ cm$

$d_m = 13.3526\ cm = 5.2569\ inches$ , $2\psi_c = 39.39^\circ$

$d_m = \sqrt{3\ell\lambda} \Rightarrow d_m^2 = 3\ell\lambda \Rightarrow \ell = d_m^2/3\lambda = (13.3526)^2/3(3) = 19.81\ cm$

$\psi_c = \sin^{-1}\left(\frac{6.676}{19.81}\right) = \sin^{-1}(0.337) = 19.695^\circ$

13-24. a. $A_{em} = (1-|\Gamma|^2)\cdot\frac{\lambda^2}{4\pi}\cdot D_0 = 0.99\cdot\frac{(0.03)^2}{4\pi}\cdot 75 = 5.317764\times10^{-3}\ m^2$

$(f = 10\,GHz \rightarrow \lambda = 0.03\,m)$

$A_{em} = 0.005317764\ m^2$

b. $P_{max} = A_{em}\cdot W_i = (1\times10^{-6}\ watts/m^2)\cdot(0.005317764) = 5.317\times10^{-9}\ watts$

$P_{max} = 5.317764\times10^{-9}\ watts$

13-25 $f = 10.3\,GHz \Rightarrow \lambda_0 = 30\times10^9/(10.3\times10^9) = 2.9126\ cm$

a. $\lambda_0/4 < d < \lambda_0/2 \Rightarrow \frac{2.9126}{4} < d < \frac{2.9126}{2} \Rightarrow 0.72185_{cm} < d < 1.4563_{cm}$

b. $W < \lambda_0/10 = 0.29126\ cm$

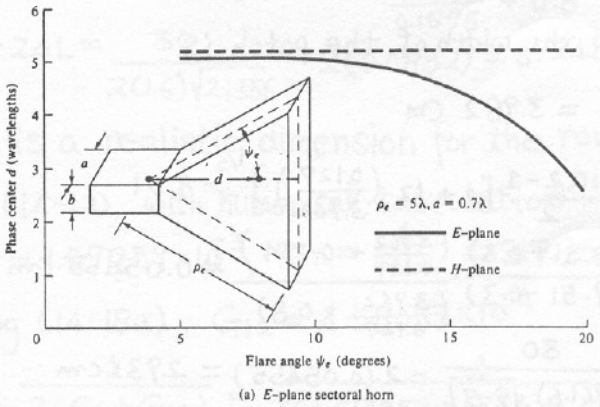
c. $t \leq W/10 = 0.29126/10 = 0.029126\ cm$

13-26 a. Using the Fig. P13.26

E-plane : $4.54\lambda$ , H-plane : $5.2\lambda$

b. Using the Fig. P13.26

E-plane : $5.06\lambda$ , H-plane : $4.5\lambda$




(a) E-plane sectoral horn

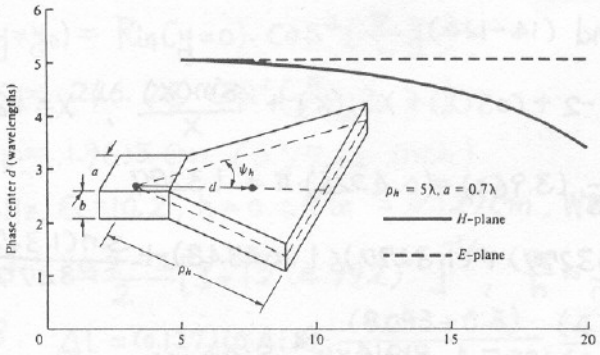



Fig. P13.26 Phase center location, as a function of flare angle, for E- and H-plane sectoral horns. (SOURCE: Y. Y. Hu, "A Method of Determining Phase Centers and Its Applications to Electromagnetic Horns," *Journal of Franklin Institute*, vol. 271, pp. 31–39, January 1961)

14-1. By using (14-1), the effective dielectric constant is equal to

$$\epsilon_{reff} = \frac{10.-1}{2} + \frac{10-1}{2}\left[1+12\cdot\left(\frac{1}{1.2}\right)\right]^{-1/2} = 6.8568$$

Using (14-19), $\frac{W}{h} = 1.2$

$$Z_c = \frac{120\cdot\pi}{\sqrt{\epsilon_{reff}}\left[1.2+1.393+0.667\cdot\ln(1.2+1.444)\right]}$$

$$= \frac{120\cdot\pi}{2.6185\ (3.24152)} = 44.415 \text{ ohms}$$

14-2. $$\epsilon_{reff} = \frac{6.8+1}{2} + \frac{6.8-1}{2}\left[1+12\left(\frac{1}{1.5}\right)\right]^{-1/2} = 4.8667$$

At low frequencies, the characteristic impedance can be found by using eq. (14-19), if $\frac{W}{h} > 1$.

$$Z_c\ (f=0) = \frac{120\pi}{\sqrt{4.8667}\cdot\left[1.5+1.393+0.667\ln(1.5+1.444)\right]}$$

$$= 170.889\ /\ (3.6132) = 47.296 \text{ ohms}$$

14-3 a. Using (14-6), the width of the patch is

$$W = \frac{30}{2(1.6)}\sqrt{\frac{2}{10.2+1}} = 3.962 \text{ cm}$$

$$\epsilon_{reff} = \frac{10.2+1}{2} + \frac{10.2-1}{2}\left[1+12\cdot\left(\frac{0.127}{3.962}\right)\right]^{-1/2} = 9.51$$

$$\Delta L = (0.127)(0.412)\frac{(9.51+0.3)}{(9.51-0.3)}\cdot\frac{\left(\frac{3962}{0.127}+0.264\right)}{\left(\frac{3.962}{0.127}+0.8\right)} = 0.05455 \text{ cm}$$

$$L = \frac{\lambda}{2} - 2\Delta L = \frac{30}{2(1.6)\sqrt{9.51}} - 2(0.05455) = 2.931 \text{ cm}$$

b. Using (14-12) and (14-12a)

$$G_{T1} = \frac{I_1}{120\pi^2}, \quad I_1 = -2 + \cos(X) + X\,Si(X) + \frac{\sin(X)}{X}, \quad X = k_0 W$$

$$X = \frac{2\pi}{\lambda_0}\cdot W = \frac{2\pi}{18.75}(3.962) = (0.4226)\cdot\pi = 1.3277$$

$$I_1 = -2 + \cos(1.3277) + (1.3277)(1.204348) + \frac{\sin(1.3277)}{1.3277}$$

$$= 0.57075$$

$$G_1 = 0.57075\ /\ (120\cdot\pi^2) = 4.81916\times10^{-4} \text{ Siemens}$$

Resonant input impedance, $R_{in} = Z_{in} = \frac{1}{2G_1} = 1.0375\times10^3 = 1037.5$ ohms

(Continued)

14-3 (Cont'd)

Direct numerical calculation from (14-12) results into

$G_1 = 4.819021 \times 10^{-4}$, $R_{in} = \frac{1}{2G_1} = 1037.56$ ohms

c. Numerical calculation from (14-18a)

$G_{12} = 3.92904 \times 10^{-4}$

d. $R_{in} = \frac{1}{2(G_1 \pm G_2)} = \frac{1}{2(4.819 + 3.929) \times 10^{-4}} = 571.56$ ohms

e. $R_{in}(y=y_0) = R_{in}(y=0)\cos^2(\frac{\pi}{L} y_0)$

$75 = 571.56 \cos^2(\frac{\pi}{2.931} \cdot y_0)$

$y_0 = 1.1197$ cm (0.4408 inch)

14-4 a. $W = \frac{30}{2(1.6)\sqrt{\frac{2}{2.2+1}}} = 7.412$ cm

$\epsilon_{reff} = \frac{2.2+1}{2} + \frac{2.2-1}{2}\left[1 + 12\left(\frac{0.1575}{7.412}\right)\right]^{-1/2} = 2.1356$

$\Delta L = (0.1575)(0.412)\frac{2.1356 + 0.3}{2.1356 - 0.258} \cdot \frac{\frac{7.412}{0.1575} + 0.264}{\frac{7.412}{0.1575} + 0.8} = 0.0832$ cm

$L = \frac{\lambda}{2} - 2\Delta L = \frac{30}{2(1.6)\sqrt{2.1356}} - 2(0.0832) = 6.2487$ cm

∴ The Patch is a realistic dimension for the roof of a personal car.

b. From (14-12), with numerical calculation

$G_1 = 1.57259 \times 10^{-3}$, $R_{in} = \frac{1}{2G_1} = 317.95$ ohms

c. Using (14-18a), $G_{12} = 4.58053 \times 10^{-4}$

d. $R_{in} = \frac{1}{2(G_1 + G_{12})} = \frac{1}{2(1.57259 \times 10^{-3} + 4.58053 \times 10^{-4})} = 246.23$ ohms

e. $R_{in}(y=y_0) = R_{in}(y=0) \cdot \cos^2(\frac{\pi}{L} y_0)$

$75 = 246.23 \cdot \cos^2(\frac{\pi}{6.2487} \cdot y_0)$

$y_0 = 1.9615$ cm (0.7722 inch)

14-5. $f_0 = 10$GHz, $\epsilon_r = 10.2$, $h = 0.05$ in $= 0.127$cm, $W = \frac{30}{2 \cdot 10} \cdot \sqrt{\frac{2}{10.2+1}} = 0.634$ cm

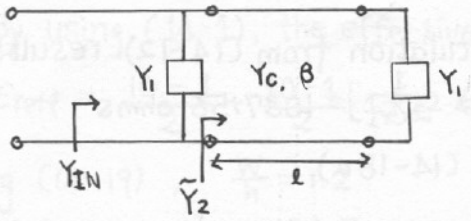
$\epsilon_{reff} = \frac{10.2+1}{2} + \frac{10.2-1}{2}\left[1 + 12(4.992)^{-1}\right]^{-1/2}$, $\frac{W}{h} = \frac{0.634}{0.127} = 4.992$

$= 8.093$. $\Delta L = (0.127)(0.412) \cdot \frac{(8.093+0.3)}{(8.093-0.258)} \frac{(4.992+0.264)}{(4.992+0.8)} = 0.0509$ cm

$L = \frac{30}{2(10)} \cdot \frac{1}{\sqrt{8.093}} - 2(0.0509) = 0.4255$ cm, $W = 0.634$ cm $= 0.2496$ in

$L = 0.4255$ cm $= 0.1675$ in

14-6.




, $Y_1 = G_1 + jB_1$

$\tilde{Y}_2 = \tilde{G}_2 + j\tilde{B}_2$

Resonance, $Im(Y_{IN}) = 0 \Rightarrow Y_{IN} = G_{IN}$

$$Y_{IN} = Y_1 + \tilde{Y}_2, \text{ where } \tilde{Y}_2 = Y_c \cdot \frac{Y_1 + jY_c \tan\beta\ell}{Y_c + jY_1 \tan\beta\ell}$$

$$\tilde{Y}_2 = Y_c \cdot \frac{G_1 + j(B_1 + Y_c \tan\beta\ell)}{(Y_c - B_1\tan\beta\ell) + jG_1\tan\beta\ell} \quad (1)$$

$$G_{IN} = Y_1 + Y_c \frac{G_1 + j(B_1 + Y_c\tan\beta\ell)}{(Y_c - B_1\tan\beta\ell) + jG_1\tan\beta\ell} \quad (2)$$

$$G_{IN}\left[(Y_c - B_1\tan\beta\ell) + jG_1\tan\beta\ell\right] = (G_1 + jB_1)\left[(Y_c - B_1\tan\beta\ell) + jG_1\tan\beta\ell\right] + Y_c\left[G_1 + j(B_1 + Y_c\tan\beta\ell)\right] \quad (3)$$

Separating into real and imaginary parts.

$$G_{IN}(Y_c - B_1\tan\beta\ell) = G_1(Y_c - B_1\tan\beta\ell) - B_1G_1\tan\beta\ell + Y_cG_1 \quad (4)$$

$$jG_{IN}\cdot G_1\tan\beta\ell = j\left[B_1(Y_c - B_1\tan\beta\ell) + G_1^2\tan\beta\ell + Y_c(B_1 + Y_c\tan\beta\ell)\right] \quad (5)$$

Rearranging the equation (4)

$$G_{IN}(Y_c - B_1\tan\beta\ell) = 2G_1(Y_c - B_1\tan\beta\ell) \Rightarrow G_{IN} = G_1 \cdot 2 \quad (6)$$

Substituting this results into (5)

$$2G_1^2\tan\beta\ell = B_1(Y_c - B_1\tan\beta\ell) + G_1^2\tan\beta\ell + Y_c(B_1 + Y_c\tan\beta\ell)$$

$$(\tan\beta\ell)\left[G_1^2 + B_1^2 - Y_c^2\right] = 2Y_cB_1, \quad \tan\beta\ell = \frac{2Y_cB_1}{B_1^2 + G_1^2 - Y_c^2}, \quad (7)$$

From $Y_{IN} = Y_1 + \tilde{Y}_2$, and $G_{IN} = 2G_1$, we can see $\tilde{G}_2 = G_1$, Since $Re(Y_1) = G_1$

Also, since imaginary part of $Y_{IN} = 0$ at resonance

$Y_{IN} = G_{IN} + jB_{IN}$, $Im(Y_{IN}) = 0$, $B_{IN} = 0$.

$$B_{IN} = B_1 + \tilde{B}_2 = 0, \quad \tilde{B}_2 = -B_1$$

$$\therefore \tilde{Y}_2 = G_1 - jB_1$$

Total input admittance is, $Y_{in} = Y_1 + \tilde{Y}_2 = 2G_1$

14-7.

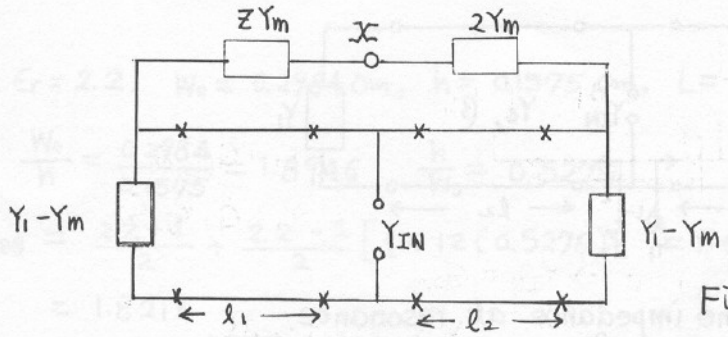
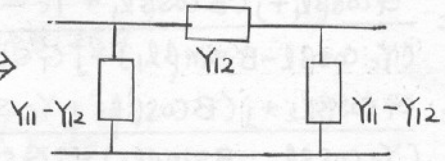


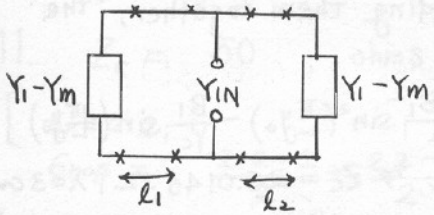

Fig. P14-7.1

This circuit has been "derived" by considering the equivalent circuit admittance for a symmetric 2-port junction.

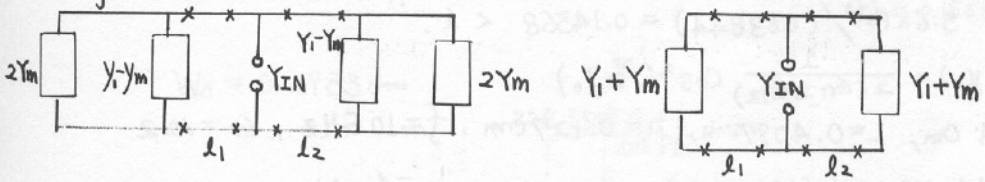
$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \Rightarrow$$




For a symmetric voltage, Node X of the Fig P14-7.1 sees an open circuit. Hence, the new circuit becomes




For an antisymmetric voltage, node X sees a zero voltage, a grounded circuit. Thus, the new circuit becomes



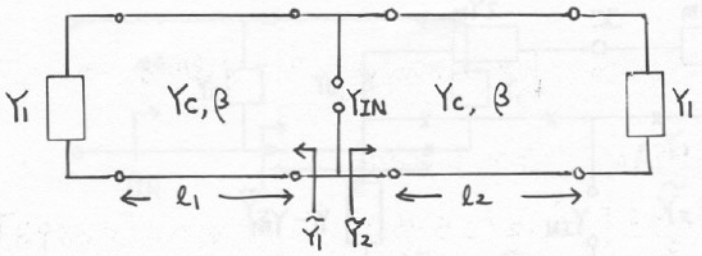

If we assume Bm to be neglible, since its effect should be to change the resonant frequency slightly, then

$$R_{in}\Big|_{Y=Y_0} \simeq \frac{\cos^2\left(\frac{\pi}{L} y_0\right)}{2(G \pm G_m)}$$

where + = odd mode

− = even mode

14-8.




We wish to find the impedance at resonance $\ell = \ell_1 + \ell_2$,

For transmission line theory $Y_{IN} = \tilde{Y}_1 + \tilde{Y}_2$,

where $\tilde{Y}_1 = Y_c \cdot \dfrac{G\cos\beta\ell_1 + j(B\cos\beta\ell_1 + Y_c \sin\beta\ell_1)}{(Y_c\cos\beta\ell - B\sin\beta\ell_1) + jG\sin\beta\ell_1}$

$\tilde{Y}_2 = Y_c \dfrac{G\cos\beta\ell_2 + j(B\cos\beta\ell_2 + Y_c\sin\beta\ell_2)}{(Y_c\cos\beta\ell_2 - B\sin\beta\ell_2) + jG\sin\beta\ell_2}$, $(\ell_1 = y_0)$ $(\ell_2 = \ell - y_0)$

At resonance, the input impedance at an arbitray feed point, a distance one end of the microstrip element, is pure real. By transforming the slot admittances $G_1 \pm jB_1$ to the common point and adding them together, the input impedance at resonance is found as

$$R_{in}(y = y_0) = \frac{1}{2(G_1 \pm G_{12})}\left[\cos^2\left(\frac{\pi}{L}y_0\right) + \frac{G_1^2 + B_1^2}{Y_c^2}\sin^2\left(\frac{\pi}{L}y_0\right) - \frac{B_1}{Y_c}\sin\left(\frac{2\pi}{L}y_0\right)\right]$$

From Example 14.2. $W = 1.186\,cm$, $h = 0.1588\,cm$, $\Rightarrow Z_c = 26.0146\,\Omega$, $\lambda = 3\,cm$.

Using (14-12) and (14-12a) $G_1 = 0.00157$ Siemens, Using (14-8b). $B_1 = 5.6\times10^{-3}$.

$\therefore (G_1^2 + B_1^2)/Y_c^2 = [(1.57\times10^{-3})^2 + (5.6\times10^{-3})^2]/[1.47\times10^{-3}] = 0.02289 \ll 1$

$\therefore B_1/Y_c = 5.6\times10^{-3}/(0.03844) = 0.14568 < 1.$

$$R_{in}(y = y_0) = \frac{1}{2(G_1 \pm G_{12})}\cos^2\left(\frac{\pi}{L}y_0\right)$$

14-9. $W = 0.634\,cm$, $L = 0.4097\,cm$, $h = 0.127\,cm$, $f = 10\,GHz$, $\epsilon_r = 10.2$

a. Using (14-12) and (14-12a), $G_1 = 4.82\times10^{-4}$ Siemens

which compares with $G_1 = 0.00175592$, $R_{in} = \dfrac{1}{2G_1} = 1037.30084$

b. $50 = 1037.30084\cdot\cos^2\left(\frac{\pi}{L}y_0\right)$

$$\cos\left(\frac{\pi}{L}y_0\right) = \sqrt{\frac{50}{1037.30084}} = 0.21955$$

$\frac{\pi}{L}y_0 = \cos^{-1}(0.21955) = 1.3494$. $\therefore y_0 = 0.1759$

14-10. $\epsilon_r = 2.2$, $W_o = 0.2984$ cm, $h = 0.1575$ cm, $L = 0.9068$ cm

$$\frac{W_o}{h} = \frac{0.2984}{0.1575} = 1.8946 \qquad \frac{h}{W_o} = 0.5278$$

$$\epsilon_{reff} = \frac{2.2+1}{2} + \frac{2.2-1}{2}\left[1 + 12(0.5278)\right]^{-1/2} = 1.6 + 0.6[0.3693]$$

$$= 1.8216$$

$$Z_c = \frac{120\pi/\sqrt{\epsilon_{reff}}}{\frac{W_o}{h} + 1.393 + 0.667 \ln\left[\frac{W_o}{h} + 1.444\right]} = \frac{120\pi/\sqrt{1.8216}}{1.8946 + 1.393 + 0.667 \ln[3.3386]}$$

$$Z_c = \frac{120\pi/(1.3497)}{4.0917} = 68.264$$

$$68.264 = 152.44 \cos^2\left(\frac{\pi}{L} y_o\right)$$

$$\cos^2\left(\frac{\pi}{L} y_o\right) = \frac{68.264}{152.44} = 0.4478$$

$$y_o = \frac{L}{\pi} \cdot \cos^{-1}(\sqrt{0.4478}) = \frac{0.9068}{\pi} \cdot (0.8377) = 0.2418 \text{ cm } (0.0952 \text{ in})$$

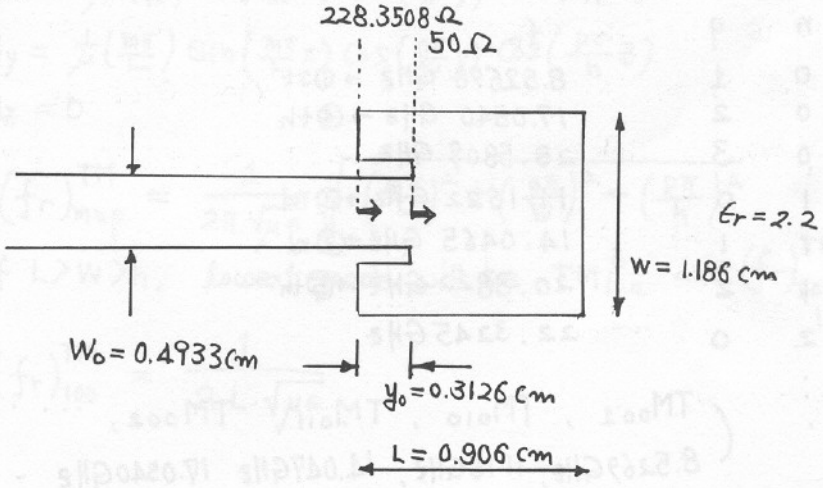
14-11. $Z_c = 50$ ohms

$\epsilon_r = 2.2$,

$$\epsilon_{reff} = \frac{2.2+1}{2} + \frac{2.2-1}{2}\left[1 + 12\frac{h}{W_o}\right]^{-1/2}$$

$$50 = \left.\frac{120 \cdot \pi}{\sqrt{\epsilon_{reff}}\left[\frac{W_o}{h} + 1.393 + 0.667 \ln\left(\frac{W_o}{h} + 1.444\right)\right]}\right|_{h = 0.1588 \text{ cm}}$$

$\therefore W_o = 0.4933$ cm.



14-12. a. 75 ohms

$$75 = 228.3508 \cos^2\left(\frac{\pi}{0.906} \cdot y_0\right), \quad \cos\left(\frac{\pi}{0.906} \cdot y_0\right) = \sqrt{\frac{75}{228.3508}}$$

$$\frac{\pi}{0.906} \cdot y_0 = \cos^{-1}(0.57309) = 0.9605, \quad y_0 = 0.277 \text{ cm}$$

Using (14-19), we can find the width of microstrip feed line by iterative method.

$$75 = \frac{120 \cdot \pi}{\sqrt{\epsilon_{reff}} \cdot \left[\frac{W}{0.1588} + 1.393 + 0.667 \ln\left(\frac{W}{0.1588} + 1.444\right)\right]}$$

∴ Dimension of line: Width : 0.2546 cm

b. 100 ohms

$$100 = 228.3508 \cos^2\left(\frac{\pi}{0.906} \cdot y_0\right), \quad \cos\left(\frac{\pi}{0.906} \cdot y_0\right) = \sqrt{\frac{100}{228.3508}}$$

$$y_0 = \frac{0.906}{\pi} \cdot \cos^{-1}(0.661757) = \frac{0.906}{\pi} \cdot (0.847636) = 0.2444 \text{ cm}$$

We can find the dimension of width W of microstrip feed line by iterative procedure as part (a) in this problem.

$$100 = \frac{120 \cdot \pi}{\sqrt{\epsilon_{reff}}\left[\frac{W}{0.1588} + 1.393 + 0.667 \ln\left(\frac{W}{0.1588} + 1.444\right)\right]}, \quad \text{if } \frac{W}{h} > 1$$

or

$$100 = \frac{60}{\sqrt{\epsilon_{reff}}} \cdot \ln\left[\frac{8h}{W} + \frac{W}{4h}\right], \quad \text{if } \frac{W}{h} \le 1$$

$$\leftarrow \epsilon_{reff} = \frac{\epsilon_r + 1}{2} + \frac{\epsilon_r - 1}{2}\left\{\left[1 + 12 \cdot \frac{h}{W}\right]^{-1/2} + 0.04\left[1 - \frac{W}{h}\right]^2\right\},$$

∴ Dimension of Width : 0.14285 cm

14-13 Using (14-31), $\epsilon_r = 2.2$, $L = 0.906$ cm, $W = 1.186$ cm

| m=0 | n | q | : $f_r$ |
|---|---|---|---|
| 0 | 0 | 1 | 8.52698 GHz → ①st |
| 0 | 0 | 2 | 17.0540 GHz → ④th |
| 0 | 0 | 3 | 25.5809 GHz |
| 0 | 1 | 0 | 11.1622 GHz → ②nd |
| 0 | 1 | 1 | 14.0465 GHz → ③rd |
| 0 | 1 | 2 | 20.3822 GHz → ⑤th |
| 0 | 2 | 0 | 22.3245 GHz |

$TM_{001}$, $TM_{010}$, $TM_{011}$, $TM_{002}$, ....

8.5269 GHz, 11.16 GHz, 14.047 GHz 17.0540 GHz ...

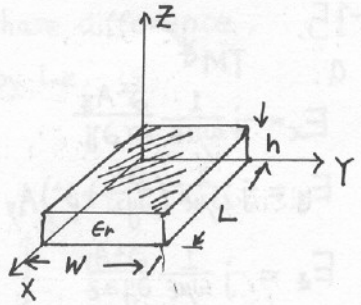
14-14.

a. $TM^z$

$$E_x = -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 A_z}{\partial x \partial z}, \qquad H_x = \frac{1}{\mu}\frac{\partial A_z}{\partial y}$$

$$E_y = -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 A_z}{\partial y \partial z}, \qquad H_y = -\frac{1}{\mu}\frac{\partial A_z}{\partial x}$$

$$E_z = -j\frac{1}{\omega\mu\epsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)A_z, \quad H_z = 0$$




$$A_z = [C_1\cdot\cos(\beta_x x)+D_1\sin(\beta_x x)]\cdot[C_2\cos(\beta_y y)+D_2\sin(\beta_y y)]\times$$
$$[C_3\cdot\cos(\beta_z\cdot z)+D_3\sin(\beta_z\cdot z)]$$

Boundary Condition

At $x=0$ and $x=L$, $H_y=0$ } PMC walls

At $y=0$ and $y=W$, $H_x=0$ }

At $z=0$ and $z=h$, $E_x=E_y=0$ } PEC walls

Applying the boundary conditions to the walls leads to

$$A_z = A_{mnp}\cos(\beta_x x)\cos(\beta_y y)\cos(\beta_z z)$$

and $\beta_x = \frac{m\pi}{L}$, $\beta_y = \frac{n\pi}{W}$, $\beta_z = \frac{p\pi}{h}$, $\begin{pmatrix} m,n,p = 0,1,2,\cdots \\ m=n=p\neq 0 \end{pmatrix}$

Thus

$$E_x = -j\frac{1}{\omega\mu\epsilon}\left(\frac{m\pi}{L}\right)\left(\frac{p\pi}{h}\right)\sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{W}y\right)\sin\left(\frac{p\pi}{h}z\right)$$

$$E_y = -j\frac{1}{\omega\mu\epsilon}\left(\frac{n\pi}{W}\right)\left(\frac{p\pi}{h}\right)\cos\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{W}y\right)\sin\left(\frac{p\pi}{h}z\right)$$

$$E_z = -j\frac{1}{\omega\mu\epsilon}\left[-\left(\frac{p\pi}{h}\right)^2+\beta^2\right]\cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{W}y\right)\cos\left(\frac{p\pi}{h}z\right)$$

$$H_x = -\frac{1}{\mu}\left(\frac{n\pi}{W}\right)\cos\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{W}y\right)\cos\left(\frac{p\pi}{h}z\right)$$

$$H_y = \frac{1}{\mu}\left(\frac{m\pi}{L}\right)\sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{W}y\right)\cos\left(\frac{p\pi}{h}z\right)$$

$$H_z = 0$$

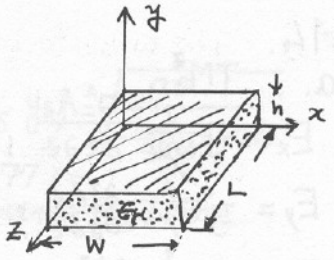
b. $$(f_r)^{TM}_{mnp} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{L}\right)^2+\left(\frac{n\pi}{W}\right)^2+\left(\frac{p\pi}{h}\right)^2}$$

c. if $L>W>h$, lowest mode will be $TM^z_{100}$ and $(f_r)^{TM}_{100} = \frac{1}{2L\sqrt{\mu\epsilon}}$

d. $$(f_r)^{TM}_{100} = \frac{1}{2L\sqrt{\mu\epsilon}}.$$

14-15.

a. $TM^y$




$$E_x = -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 A_y}{\partial x \partial y} \qquad H_x = -\frac{1}{\mu}\frac{\partial A_y}{\partial z}$$

$$E_y = j\frac{1}{\omega\mu\epsilon}\left(\frac{\partial^2}{\partial y^2}+\beta^2\right)A_y \qquad H_y = 0$$

$$E_z = -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 A_y}{\partial y \partial z} \qquad H_z = \frac{1}{\mu}\frac{\partial A_y}{\partial x}$$

$$A_y = [C_1\cos(\beta_x x)+D_1\sin(\beta_x x)][C_2\cos(\beta_y y)+D_2\sin(\beta_y y)][C_3\cos(\beta_z z)+D_3\sin(\beta_z z)]$$

Boundary Condition

At $x=0$ and $x=W$, $H_z = 0$ → PMC walls

At $y=0$ and $y=h$, $E_x = E_z = 0$ → PEC walls

At $z=0$ and $z=L$, $H_x = 0$ → PMC walls

Applying the boundary conditions to the walls leads to

$$A_y = A_{mnp}\cos(\beta_x x)\cos(\beta_y y)\cos(\beta_z z)$$

and $\beta_x = \frac{m\pi}{W}$, $\beta_y = \frac{n\pi}{h}$, $\beta_z = \frac{p\pi}{L}$, $\begin{pmatrix} m,n,p = 0,1,2,\cdots \\ m=n=p\neq 0. \end{pmatrix}$

Thus

$$E_x = -j\frac{1}{\omega\mu\epsilon}\left(\frac{m\pi}{W}\right)\left(\frac{n\pi}{h}\right)\sin\left(\frac{m\pi}{W}x\right)\sin\left(\frac{n\pi}{h}y\right)\cos\left(\frac{p\pi}{L}z\right)$$

$$E_y = j\frac{1}{\omega\mu\epsilon}\left[-\left(\frac{n\pi}{h}\right)^2+\beta^2\right]\cos\left(\frac{m\pi}{W}x\right)\cos\left(\frac{n\pi}{h}y\right)\cos\left(\frac{p\pi}{L}z\right)$$

$$E_z = -j\frac{1}{\omega\mu\epsilon}\left(\frac{n\pi}{h}\right)\left(\frac{p\pi}{L}\right)\cos\left(\frac{m\pi}{W}x\right)\sin\left(\frac{n\pi}{h}y\right)\sin\left(\frac{p\pi}{L}z\right)$$

$$H_x = -\frac{1}{\mu}\left(\frac{p\pi}{L}\right)\cos\left(\frac{m\pi}{W}x\right)\cos\left(\frac{n\pi}{h}y\right)\sin\left(\frac{p\pi}{L}z\right)$$

$$H_y = 0$$

$$H_z = -\frac{1}{\mu}\left(\frac{m\pi}{W}\right)\sin\left(\frac{m\pi}{W}x\right)\cos\left(\frac{n\pi}{h}y\right)\cos\left(\frac{p\pi}{L}z\right)$$

b. $$(f_r)_{mnp}^{TM} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{W}\right)^2+\left(\frac{n\pi}{h}\right)^2+\left(\frac{p\pi}{L}\right)^2}$$

c. if $L>W>h$, lowest model will be $TM^y_{001}$

d. $$(f_r)_{001}^{TM^y} = \frac{1}{2L\sqrt{\mu\epsilon}}$$

14-16 The array factor of 2 elements of zero phase difference, placed along the y-direction and separated by $L_e$, is

$$(AF)_2 = e^{j\frac{k_0 L_e}{2}\cos\gamma} + e^{-j\frac{k_0 L_e}{2}\cos\gamma}$$

$$= e^{j\frac{k_0 L_e}{2}\sin\theta\sin\phi} + e^{-j\frac{k_0 L_e}{2}\sin\theta\sin\phi}$$

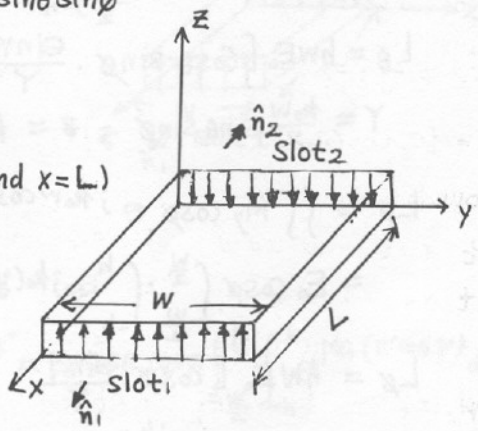
$$(AF)_2 = 2\cos\left(\frac{k_0 L_e}{2}\sin\theta\sin\phi\right)$$

14-17. At the two radiating edges ($x=0$ and $x=L$)

$E_z(x=L) = E_0$ , $H_y(x=L) = 0$

$E_z(x=0) = -E_0$ , $H_y(x=0) = 0$

We have constant distribution of $E_z$ and zero distribution of $H_y$.




Thus, on slot 1 ($x=L$)

$$\underline{M}_1 = -\hat{n}_1 \times [E_z(x=L)\hat{a}_z] = -\hat{a}_x \times E_0 \hat{a}_z = E_0 \hat{a}_y \quad , \quad M_1 = E_0 \hat{a}_y$$

$$\underline{J}_1 = \hat{n}_1 \times [H_y(x=L)\hat{a}_y] = 0 \quad , \quad J_1 = 0$$

Similarly, on slot 2, we have

$$\underline{M}_2 = -\hat{n}_2 \times [-E_0 \hat{a}_z] = E_0 \hat{a}_y$$

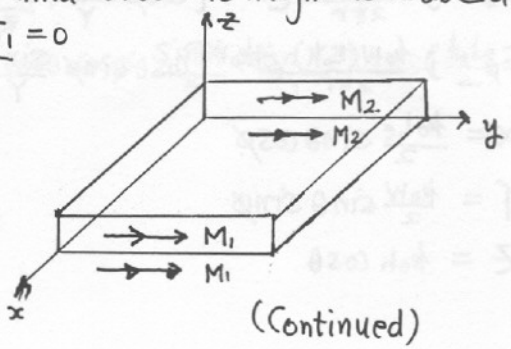
$$\underline{J}_2 = 0$$

since $\underline{M}_1$ and $\underline{M}_2$ are parallel to the PEC ground plane, their images below the plane are identical to those above the plane. In addition, $\underline{M}_1$ and $\underline{M}_2$ are constant over the apertures. Thus, removing the ground plane is equal to double the height of the slots.

To find the radiated field by the two slots, first integrate over slot 1, and then use array factor to combine the contribution from both slots.

a. On slot 1, keeping in mind that the height is doubled

$$\underline{M}_1 = E_0 \hat{a}_y \quad , \quad \underline{J}_1 = 0$$




(Continued)

14-17. (Cont'd)

then $L_\theta = \iint M_y \cos\theta \sin\phi \, e^{+jk_o r' \cos\psi} ds'$

$$= E_o \cos\theta \sin\phi \int_{-\frac{W}{2}}^{\frac{W}{2}} \int_{-h}^{h} e^{jk_o(y'\sin\theta\sin\phi + z'\cos\theta)} dz'dy'$$

$$L_\theta = hWE_o \left[\cos\theta \sin\phi \cdot \frac{\sin Y}{Y} \cdot \frac{\sin z}{z}\right]$$

$$Y = \frac{k_o W}{2} \sin\theta \sin\phi \;, \quad z = k_o h \cos\theta$$

$$L_\phi = \iint M_y \cos\phi \, e^{jk_o r' \cos\psi} ds'$$

$$= E_o \cos\phi \int_{-\frac{W}{2}}^{\frac{W}{2}} \int_{-h}^{h} e^{jk_o(y'\sin\theta\sin\phi + z'\cos\theta)} dz'dy'$$

$$L_\phi = hWE_o \left[\cos\phi \cdot \frac{\sin Y}{Y} \cdot \frac{\sin z}{z}\right]$$

thus $E_\theta \approx -j \frac{k_o e^{-jk_o r}}{4\pi r} (L_\phi + \eta_o \cancel{N_\theta}^{\,0})$

$$E'_\theta \approx -j \frac{hWE_o k_o}{4\pi r} e^{-jk_o r} \left[\cos\phi \frac{\sin Y}{Y} \cdot \frac{\sin z}{z}\right]$$

$$E_\phi \approx +j \frac{k_o e^{-jk_o r}}{4\pi r} (L_\theta - \eta_o N_\phi)$$

$$E'_\phi \approx +j \frac{hWE_o k_o}{4\pi r} e^{-jk_o r} \left[\cos\theta \sin\phi \frac{\sin Y}{Y} \cdot \frac{\sin z}{z}\right]$$

b. The array factor of 2 elements of zero phase difference, placed along the x-direction and seperated by $L_e$, is

$$(AF)_2 = e^{j\frac{k_o L_e}{2}\cos\gamma} + e^{-j\frac{k_o L_e}{2}\cos\gamma}$$

$$= e^{j\frac{k_o L_e}{2}\sin\theta\cos\phi} + e^{-j\frac{k_o L_e}{2}\sin\theta\cos\phi}$$

c. $(AF)_2 = 2 \cdot \cos\left(\frac{k_o L_e}{2} \sin\theta\cos\phi\right)$,

Finally, the field radiated by both slots is $\underline{E}^t = \underline{E}' \cdot (AF)_2$ that is

$$E_\theta = -j \frac{k_o W(E_o h)}{2\pi r} e^{-jk_o r} \left[\cos\phi \frac{\sin Y}{Y} \cdot \frac{\sin z}{z} \cdot \cos X\right]$$

$$E_\phi = j \frac{k_o W(E_o h)}{2\pi r} e^{-jk_o r} \left[\cos\theta \sin\phi \frac{\sin Y}{Y} \cdot \frac{\sin z}{z} \cdot \cos X\right]$$

$$X = \frac{k_o L_e}{2} \sin\theta\cos\phi$$

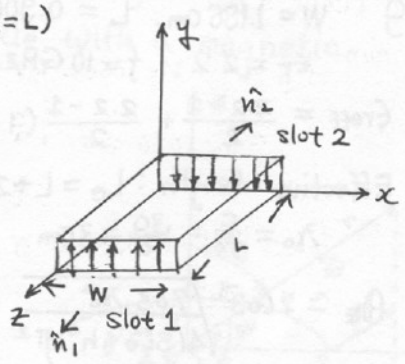
$$Y = \frac{k_o W}{2} \sin\theta \sin\phi$$

$$Z = k_o h \cos\theta$$

14-18. At the two radiating edges ($z=0$ & $z=L$)

$E_y(z=L)=E_0$, $H_x(z=L)=0$

$E_y(z=0)=-E_0$, $H_x(z=0)=0$




a. Thus, on slot 1 ($z=L$)

$\underline{M}_1 = -\hat{n}_1 \times [E_y(z=L)\hat{a}_y] = -\hat{a}_z \times E_0\hat{a}_y = E_0\hat{a}_x$

$\underline{J}_1 = 0$

similarly, on slot 2, we have

$\underline{M}_2 = E_0\hat{a}_x$

$\underline{J}_2 = 0$

$$L_\theta = \iint M_x \cos\theta\cos\phi\, e^{jk_0 r'\cos\psi}\, ds' = E_0\cos\theta\cos\phi \int_{-\frac{W}{2}}^{\frac{W}{2}}\int_{-h}^{h} e^{jk_0(r'\cos\psi)}\, dy\, dx$$

$$= E_0\cos\theta\cos\phi \int_{-\frac{W}{2}}^{\frac{W}{2}}\int_{-h}^{h} e^{jk_0(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)}\, dy\, dx$$

$$L_\theta = hWE_0\left[\cos\theta\cos\phi\ \frac{\sin X}{X}\cdot\frac{\sin Y}{Y}\right], \quad X = \frac{k_0 W}{2}\sin\theta\cos\phi,\ Y = k_0 h\sin\theta\sin\phi$$

$$L_\phi = \iint -M_x\cdot\sin\phi\, e^{jk_0 r'\cos\psi}\, ds' = hWE_0\left[\sin\phi\cdot\frac{\sin X}{X}\cdot\frac{\sin Y}{Y}\right]$$

Thus $$E_\theta' \approx +j\frac{hWE_0k_0}{4\pi r}\cdot e^{-jk_0 r}\left[\sin\phi\cdot\frac{\sin X}{X}\cdot\frac{\sin Y}{Y}\right]$$

$$E_\phi' \approx +j\frac{hWE_0k_0}{4\pi r}\cdot e^{-jk_0 r}\left[\cos\theta\cdot\cos\phi\cdot\frac{\sin X}{X}\cdot\frac{\sin Y}{Y}\right]$$

b. $$(AF)_2 = e^{j\frac{k_0 L_e}{2}\cos\gamma} + e^{-j\frac{k_0 L_e}{2}\cos\gamma}$$

$$= 2\cos\left(\frac{k_0 L_e}{2}\cos\theta\right)$$

c. Finally, the field radiated by both slots is $\underline{E}^t = \underline{E}'\cdot(AF)_2$

$$E_\theta = j\frac{k_0 hWE_0}{2\pi r}\cdot e^{-jk_0 r}\left[\sin\phi\cdot\frac{\sin X}{X}\cdot\frac{\sin Y}{Y}\right]\cdot\cos\left(\frac{k_0 L_e}{2}\cdot\cos\theta\right)$$

$$E_\phi = j\frac{k_0 hWE_0}{2\pi r}\cdot e^{-jk_0 r}\left[\cos\theta\cdot\cos\phi\ \frac{\sin X}{X}\cdot\frac{\sin Y}{Y}\right]\cos\left(\frac{k_0 L_e}{2}\cdot\cos\theta\right)$$

$$X = \frac{k_0 W}{2}\sin\theta\cos\phi$$

$$Y = k_0 h\cdot\sin\theta\cdot\sin\phi$$

14-19. $W = 1.186\,cm$, $L = 0.906\,cm$, $h = 0.1588\,cm$

$\epsilon_r = 2.2$, $f = 10\,GHz$.

$$\epsilon_{reff} = \frac{2.2+1}{2} + \frac{2.2-1}{2}\left(1+12\frac{0.1588}{1.186}\right)^{-1/2} = 1.972$$

Effective length: $L_e = L + 2\Delta L = \frac{\lambda}{2} = \frac{30}{2(10)\sqrt{1.972}} = 1.068\,cm$

$$\lambda_0 = \frac{c}{f_0} = \frac{30}{10} = 3\,cm$$

$$\theta_E \simeq 2\cos^{-1}\sqrt{\frac{7.03\lambda_0^2}{4(3L_e^2+h^2)\pi^2}} \simeq 2\cos^{-1}(0.4649) = 2(1.0872) = 2.1745\,rad = 124.59^\circ$$

$$\theta_H \simeq 2\cos^{-1}\sqrt{\frac{1.}{2+k_0W}} \simeq 2\cos^{-1}\sqrt{\frac{1.}{2+\frac{2\pi}{3}(1.186)}} = 2\cos^{-1}(0.472247)$$

$$= 2(1.0789) = 2.1578\,rad = 123.6328^\circ$$

a. $$D_0 = \frac{4\pi}{\theta_E\theta_H} = \frac{4\pi}{(2.1745)(2.1578)} = 2.6782 = 4.278\,dB$$

b. $$D_0 = \frac{22.181}{\theta_E^2+\theta_H^2} = \frac{22.181}{(2.1745)^2+(2.1578)^2} = 2.3638 = 3.736\,dB$$

14-20. $g_{12} = \frac{G_{12}}{G_1} = \frac{3.92904\times10^{-4}}{4.819021\times10^{-4}} = 0.8153$, $D_{AF} = \frac{2}{1+g_{12}} = \frac{2}{1+0.8153} = 1.101746$

Using (14-53) and (14-53a)

$I_1 = 0.57074$

$$D_0 = \left(\frac{2\pi\cdot 3.962}{18.75}\right)^2\cdot\left(\frac{1}{0.57074}\right) = (1.32768)^2\cdot(1.752) = 3.0885 = 4.897\,dB$$

$\therefore D_2 = D_0\cdot D_{AF} = (3.0885)(1.101746) = 3.4027 = 5.32\,dB$

Using (14-55a), $I_2 = 1.62558$

Finally using (14-55)

$$D_2 = \left(\frac{2\pi W}{\lambda_0}\right)^2\cdot\frac{\pi}{I_2} = \left((1.32768)^2\cdot(\pi/1.62558)\right) = 3.4066 = 5.32\,dB$$

14-21. $g_{12} = \frac{G_{12}}{G_1} = \frac{4.58053\times10^{-4}}{1.57259\times10^{-3}} = 0.2912878$, $D_{AF} = \frac{2}{1+g_{12}} = 1.5488486$

Using (14-53) and (14-53a), $I_1 = 1.8625$,

$$D_0 = \left(\frac{2\pi\cdot 7.412}{18.75}\right)^2\cdot\frac{1}{1.8625} = 3.3123 = 5.20\,dB$$

$\therefore D_2 = D_0\cdot D_{AF} = (3.3123)(1.5488486) = 5.130 = 7.1\,dB$

Using (14-55a), $I_2 = 6.4152$

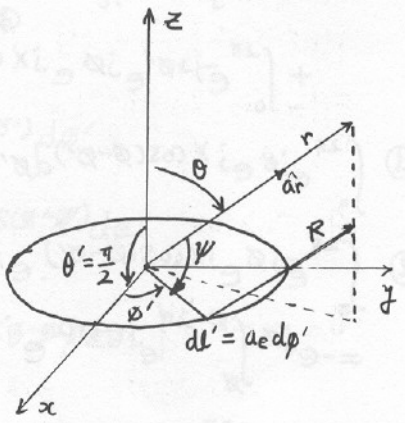
Finally Using (14-55)

$$D_2 = \left(\frac{2\pi W}{\lambda_0}\right)^2\cdot\frac{\pi}{I_2} = \left(\frac{2\pi(7.412)}{18.75}\right)^2\cdot\frac{\pi}{6.4152} = 3.0211 = 4.8016\,dB$$

14-22. Using the equivalence principle the cavity can be modeled as a circular loop antenna of radius $a_e$ with a magnetic current of

$$I_m = \hat{a}_\phi \, 2V_0 \cos\phi'$$

where at $\phi'=0$, $V_0 = hE_0 J_1(ka_e)$




$$E_\theta \simeq -j\frac{ke^{-jkr}}{4\pi r}(L_\phi + \eta N_\theta)$$

$$E_\phi \simeq +j\frac{ke^{-jkr}}{4\pi r}(L_\theta - \eta N_\phi) \quad \text{but } N_\theta = N_\phi = 0$$

∴ Our task then is to find $L_\phi$ and $L_\theta$

$$L_\theta = \iint_s M_\phi \cos\theta \sin(\phi-\phi') e^{jkr'\cos\psi} ds'$$

where $r'\cos\psi = \rho'\sin\theta\cos(\phi-\phi') = a_e \cdot \sin\theta \cdot \cos(\phi-\phi')$

$\rho' = a_e$ because the current source exist only at $\rho' = a_e$.

and $ds' = \rho' d\rho' d\phi' = a_e d\phi'$

$$L_\theta = \int_0^{2\pi} M_\phi \cos\theta \cdot \sin(\phi-\phi') e^{jka_e \sin\theta\cos(\phi-\phi')} a_e d\phi'$$

$$= \int_0^{2\pi} 2V_0 a \cos\phi' \cos\theta \sin(\phi-\phi') e^{jX\cos(\phi-\phi')} d\phi'$$

$$= 2V_0 a \cos\theta \int_0^{2\pi} \cos\phi' \cdot \sin(\phi-\phi') e^{jX\cos(\phi-\phi')} d\phi'$$

where $X = k a_e \sin\theta$, use $\cos\phi' = \frac{e^{j\phi'}+e^{-j\phi'}}{2}$, $\sin(\phi-\phi') = \frac{e^{j(\phi-\phi')} - e^{-j(\phi-\phi')}}{2j}$

substituting

$$L_\theta = 2V_0 a_e \cos\theta \int_0^{2\pi} \frac{1}{4j}(e^{j\phi'}+e^{-j\phi'})(e^{j(\phi-\phi')} - e^{-j(\phi-\phi')}) e^{jX\cos(\phi-\phi')} d\phi'$$

$$= \frac{V_0 a_e \cos\theta}{2j} \int_0^{2\pi} \left[e^{j\phi'}e^{j(\phi-\phi')} - e^{j\phi'}e^{-j(\phi-\phi')} + e^{-j\phi'}e^{j(\phi-\phi')} - e^{-j\phi'-j(\phi-\phi')}\right] \times \cdot e^{+jX\cos(\phi-\phi')} d\phi'$$

$$= \frac{V_0 a_e \cos\theta}{2j} \int_0^{2\pi} \left[e^{j\phi} - e^{j(2\phi'-\phi)} + e^{-j(2\phi'-\phi)} - e^{-j\phi}\right] e^{jX\cos(\phi-\phi')} d\phi'$$

(Continued)

14-22 (Cont'd)

$$L_\theta = \frac{V_0 a_e \cos\theta}{2j}\Big[\underbrace{\int_0^{2\pi} e^{j\phi} e^{jX\cos(\phi-\phi')}d\phi'}_{①} - \underbrace{\int_0^{2\pi} e^{j2\phi'-j\phi} e^{jX\cos(\phi-\phi')}d\phi'}_{②}$$
$$+ \underbrace{\int_0^{2\pi} e^{-j2\phi'} e^{j\phi} e^{jX\cos(\phi-\phi')}d\phi'}_{③} - \underbrace{\int_0^{2\pi} e^{-j\phi} e^{jX\cos(\phi-\phi')}d\phi'}_{④}\Big]$$

① $\int_0^{2\pi} e^{j\phi} e^{jX\cos(\phi-\phi')}d\phi' = e^{j\phi}\, 2\pi\cdot J_0(X)$

② $\int_0^{2\pi} e^{-j\phi} e^{jX\cos(\phi-\phi')}\cdot e^{j2\phi'}d\phi' = e^{j\phi}\int_0^{2\pi} e^{j2(\phi'-\phi)} e^{jX\cos(\phi-\phi')}d\phi'$

$= -e^{j\phi}\int_{\phi}^{\phi-2\pi} e^{j(-2)u}\cdot e^{jX\cos u}\,du$ , $\langle \leftarrow u=\phi-\phi',\ du=-d\phi' \rangle$

$= e^{j\phi}\int_0^{2\pi} e^{j(-2)u} e^{jX\cos u}\,du = e^{j\phi}\left[\frac{2\pi}{j^{-n}} J_n(X)\right]_{n=-2} = -2\pi e^{j\phi}\cdot J_{-2}(X)$

$= -2\pi e^{j\phi}(-1)^2 J_2(X) = -2\pi e^{j\phi} J_2(X)$

③ $\int_0^{2\pi} e^{-j2\phi'} e^{j\phi} e^{jX\cos(\phi-\phi')}d\phi' = e^{-j\phi}\int_0^{2\pi} e^{j2(\phi-\phi')} e^{jX\cos(\phi-\phi')}d\phi'$

$= -e^{-j\phi}\cdot\int_{\phi-0}^{\phi-2\pi} e^{j2u}\cdot e^{jX\cos u}\,du = e^{j\phi}\int_0^{2\pi} e^{j2u} e^{jX\cos u}\,du = e^{-j\phi}\left[\frac{2\pi}{j^{-n}} J_n(X)\right]_{n=2}$

$= -e^{-j\phi}\cdot 2\pi\cdot J_2(X)$

④ $\int_0^{2\pi} e^{-j\phi} e^{jX\cos(\phi-\phi')}d\phi' = e^{-j\phi}\, 2\pi\cdot J_0(X)$

$$\therefore L_\theta = \frac{V_0 a_e\cos\theta}{2j}\left[e^{j\phi}2\pi J_0(X) + 2\pi e^{j\phi} J_2(X) - 2\pi e^{-j\phi} J_2(X) - 2\pi e^{-j\phi} J_0(X)\right]$$

$$= \frac{2\pi V_0 a_e\cos\theta}{2j}\left[J_0(X)(e^{j\phi} - e^{-j\phi}) + J_2(X)(e^{j\phi} - e^{-j\phi})\right]$$

$$L_\theta = \frac{2\pi V_0 a_e\cos\theta}{2j}\, 2j\sin\phi\,[J_0(X)+J_2(X)] = 2\pi V_0 a_e\cos\theta\sin\phi\,[J_0(X)+J_2(X)]$$

$$\therefore E_\phi \approx jk\frac{e^{-jkr}}{2r} V_0 a_e\cdot\cos\theta\cdot\sin\phi\,[J_0(ka_e\sin\theta) + J_2(ka_e\sin\theta)]$$

(Continued)

14-22 (Cont'd)

$$L_\phi = \int_0^{2\pi} 2V_0 a_e \cos\phi' \cos(\phi-\phi') e^{jka_e \sin\theta \cos(\phi-\phi')} d\phi'$$

$$= 2V_0 a_e \int_0^{2\pi} \cos\phi' \cdot \cos(\phi-\phi') e^{jX\cos(\phi-\phi')} d\phi'$$

$$= 2V_0 a_e \int_0^{2\pi} \frac{e^{j\phi'}+e^{-j\phi'}}{2} \cdot \frac{e^{j(\phi-\phi')}+e^{-j(\phi-\phi')}}{2} e^{jX\cos(\phi-\phi')} d\phi'$$

$$= \frac{V_0 a_e}{2}\left[\int_0^{2\pi} e^{j\phi} e^{jX\cos(\phi-\phi')} d\phi' + \int_0^{2\pi} e^{j2\phi'-j\phi} e^{jX\cos(\phi-\phi')} d\phi' \right.$$

$$\left. + \int_0^{2\pi} e^{-j2\phi'} e^{j\phi} e^{jX\cos(\phi-\phi')} d\phi' + \int_0^{2\pi} e^{-j\phi} e^{jX\cos(\phi-\phi')} d\phi' \right]$$

① ② ③ ④

① $\int_0^{2\pi} e^{j\phi} e^{jX\cos(\phi-\phi')} d\phi' = 2\pi e^{j\phi} J_0(X)$

② $\int_0^{2\pi} e^{-j\phi} e^{jX\cos(\phi-\phi')} d\phi' = 2\pi e^{-j\phi} J_0(X)$

③ $\int_0^{2\pi} e^{-j\phi} e^{j2\phi'} e^{jX\cos(\phi-\phi')} d\phi' = -2\pi e^{j\phi} J_2(X)$

④ $\int_0^{2\pi} e^{j\phi} e^{-j2\phi'} e^{jX\cos(\phi-\phi')} d\phi' = -2\pi e^{-j\phi} J_2(X)$

$$L_\phi = \frac{V_0 a_e}{2}\left[2\pi J_0(X) e^{j\phi} + 2\pi J_0(X) e^{-j\phi} - 2\pi J_2(X) e^{j\phi} - 2\pi J_2(X) e^{-j\phi}\right]$$

$$L_\phi = 2\pi V_0 a_e \cos\phi \left[J_0(X) - J_2(X)\right]$$

$$\therefore E_\theta \simeq \frac{-jke^{-jkr}}{4\pi r} 2\pi V_0 a_e \cos\phi \left[J_0(X) - J_2(X)\right]$$

$$E_\theta \simeq -jk \frac{e^{-jkr}}{2r} V_0 a_e \cos\phi \left[J_0(ka\sin\theta) - J_2(ka\sin\theta)\right]$$

The $E_\theta$ and $E_\phi$ components of the far-field are given as

$$E_\theta \simeq -j \frac{k a_e V_0 e^{-jkr}}{2r} \{\cos\phi \, J_{02}'\}$$

$$E_\phi \simeq j k \frac{a_e V_0 e^{-jkr}}{2r} \{\cos\theta \sin\phi \, J_{02}\}$$

$$J_{02}' = J_0(ka_e\sin\theta) - J_2(ka_e\sin\theta)$$

$$J_{02} = J_0(ka_e\sin\theta) + J_2(ka_e\sin\theta)$$

14-23. $A_z = B_{mnp} J_m(k_\rho \rho)[A_2 \cos(m\phi) + B_2 \sin(m\phi)] \cos(k_z z)$

Boundary Condition,

$H_\phi (\rho = a, 0 \le \phi \le \pi, 0 \le z \le h) = 0$

$H_\rho (0 \le \rho \le a, \phi = 0, \text{or } \phi = \pi, 0 \le z \le h) = 0$

$H_\rho(\phi=0) = \frac{1}{\mu}\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} = \frac{1}{\mu}\frac{1}{\rho} \cdot B_{mnp} \cdot J_m(k_\rho \rho)[-A_2 \sin(m\phi)\cdot m + mB_2 \cdot \cos(m\phi)] \cos k_z z$

$\therefore B_2 = 0,$

$H_\rho(\phi=\pi) = \frac{1}{\mu}\frac{1}{\rho} B_{mnp} \cdot J_m(k_\rho \rho)[-A_2 \sin(m\phi)\cdot m] \cos k_z z = 0. \quad m\pi = \sin^{-1}(0) = p\pi$

$m = p = 1, 2, 3, 4 \cdots$

$H_\phi(\rho=a) = -\frac{1}{\mu}\frac{\partial A_z}{\partial \rho} = -\frac{1}{\mu} \cdot B_{mnp} J_m'(k_\rho a)[A_2 \cos(m\phi)] \cos(k_z z) = 0.$

$J_m'(k_\rho a) = 0, \quad k_\rho = \chi_{mn}'/a,$

From the Boundary Condition $E_\rho(0 \le \rho \le a, 0 \le \phi \le \pi, z=0) = E_\rho(0 \le \rho \le a, 0 \le \phi \le \pi, z=h) = 0.$

$E_\phi(0 \le \rho \le a, 0 \le \phi \le \pi, z=0) = E_\phi(0 \le \rho \le a, 0 \le \phi \le \pi, z=h) = 0.$

$k_z = \frac{p\pi}{h}. \quad \therefore m = 1, 2, 3, 4, \cdots, \; n = 1, 2, 3, 4 \cdots, p = 0, 1, 2, 3, 4 \cdots$

Therefore the resonant frequencies for the $TM^z_{mnp}$ modes can be written

$2\pi f_r \sqrt{\mu\epsilon} = \omega_r \sqrt{\mu\epsilon} = \beta_r = \sqrt{\beta_\rho^2 + \beta_z^2} \qquad (\leftarrow k_\rho = \beta_r, \; k_z = \beta_z)$

$f_r = \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{\chi_{mn}'}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2}, \quad m = 1, 2, 3 \cdots, \; n = 1, 2, 3 \cdots, \; p = 0, 1, 2, 3, \cdots$

14-24. The solution to this problem is identical to that of Problem 14-23 with exceptions of m.

$TM^z$: $\frac{m\pi}{2} = \sin^{-1}(0) = q\pi, \quad q = 1, 2, 3, 4 \cdots, \; m = 2, 4, 6.$

$f_r = \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{\chi_{mn}'}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2}$

$m = 2, 4, 6, 8, \cdots$

$n = 1, 2, 3, \cdots$

$p = 0, 1, 2, 3 \cdots$

14-25. Without considering the feed point, we can use cavity model. The solution to this problem is identical to that of Problem 14-23, with exceptions of m , $TM^{z}_{mnp}$.

$$H_\phi(\phi=\phi_0) = \frac{1}{\mu}\frac{1}{\rho} B_{mnp} J_m(k_\rho \rho)[-A_2 \cdot m \cdot \sin(m\phi_0)] \cos k_z z = 0,$$

$$\therefore m\phi_0 = \sin^{-1}(0) = q\pi.$$

$$\therefore m = \frac{q\pi}{\phi_0}, \quad q = 1, 2, 3, 4, 5, \cdots$$

$$f_r = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{\chi_{mn}}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2}, \quad m = \frac{q\pi}{\phi_0}, \; q = 1, 2, 3, \cdots$$

$$n = 1, 2, 3, \cdots$$

$$p = 0, 1, 2, 3, \cdots$$

14-26. If we use the cavity model, $A_z$ can be written as

$$A_z(\rho, \phi, z) = [A_1 J_m(k_\rho \rho) + B_1 Y_m(k_\rho \rho)][C_2 \cos m\phi + D_2 \sin m\phi][C_3 \cos k_z z]$$

Applying the boundary condition leads to

$$H_\phi(\rho = a, \; 0 \le \phi < 2\pi, \; 0 \le z \le h) = -\frac{1}{\mu}\frac{\partial A_z}{\partial \rho}$$

$$= -\frac{1}{\mu}[A_1 J_m'(k_\rho a) + B_1 Y_m'(k_\rho a)][C_2 \cos m\phi + D_2 \sin m\phi][C_3 \cos k_z z] = 0$$

$$H_\phi(\rho = b, \; 0 \le \phi \le 2\pi, \; 0 \le z \le h)$$

$$= -\frac{1}{\mu}[A_1 J_m'(k_\rho b) + B_1 Y_m'(k_\rho b)][C_2 \cos m\phi + D_2 \sin m\phi][C_3 \cos k_z z] = 0.$$

$$E_\phi(a \le \rho \le b, \; 0 \le \phi \le 2\pi, \; z = 0, \text{ and } z = h) = 0.$$

$$= -j\frac{1}{\omega\mu\epsilon}\frac{1}{\rho}\frac{\partial^2 A_z}{\partial \phi \, \partial z} \rightarrow \quad k_z = \frac{p\pi}{h}, \quad p = 0, 1, 2, \cdots$$

$$\begin{bmatrix} J_m'(k_\rho a) & Y_m'(k_\rho a) \\ J_m'(k_\rho b) & Y_m'(k_\rho b) \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \therefore J_m'(k_\rho a) Y_m'(k_\rho b) - J_m'(k_\rho b) Y_m'(k_\rho a) = 0$$

To find the resonant frequency for $TM^{z}_{mnp}$ mode, we must solve the following equation.

$$J_m'(k_\rho a) Y_m'(k_\rho b) - J_m'(k_\rho b) Y_m'(k_\rho a) = 0$$

$k_{mn}$ must satisfy the characteristic equation.

14-27. The solution to this problem is identical to that of Problem 14-26, with following exceptions;

$$E_z = -j\frac{1}{\omega\mu\epsilon}\left(\frac{\partial^2}{\partial z^2}+k^2\right)A_z = -j\frac{k_\rho^2}{\omega\mu\epsilon}\left[A_1 J_m(k_\rho\rho)+B_1 Y_m(k_\rho\rho)\right]\left[C_2\cos(m\phi)+D_2\sin m\phi\right]\times \left[C_3\cos k_z z\right].$$

$$E_z(\phi=0) = -j\frac{k_\rho^2}{\omega\mu\epsilon}\left[A_1 J_m(k_\rho\rho)+B_1 Y_m(k_\rho\rho)\right]\left[C_2(1)+D_2(0)\right]C_3\cos k_z z = 0.$$

$$\Rightarrow \quad \therefore C_2 = 0$$

$$E_z(\phi=\phi_0) = -j\frac{k_\rho^2}{\omega\mu\epsilon}\left[A_1 J_m(k_\rho\rho)+B_1 Y_m(k_\rho\rho)\right]\left[D_2\sin(m\phi_0)\right]C_3\cos k_z z = 0$$

$$\Rightarrow \sin(m\phi_0) = 0$$

$$m\phi_0 = \sin^{-1}(0) = p\pi \Rightarrow m = \frac{p\pi}{\phi_0}, \quad p = 1, 2, 3, 4, \cdots$$

$$J_m'(k_\rho a)\,Y_m'(k_\rho b) - J_m'(k_\rho b)\,Y_m'(k_\rho a) = 0$$

14-28. The designed center frequency $\Rightarrow$ 1.6 GHz
the dielectric constant of the substrate $\Rightarrow$ 10.2 (i.e, RT/duroid)
$h = 0.127$ cm

Using (14-69a) $$F = \frac{8.791}{1.6\sqrt{10.2}} = 1.7203555$$

Therefore using (14-69)

$$a = \frac{F}{\left\{1+\frac{2h}{\pi\epsilon_r F}\left[\ln\left(\frac{\pi F}{2h}\right)+1.7726\right]\right\}^{1/2}} = \frac{1.7203555}{\left\{1+0.0222555\right\}^{1/2}} = 1.701525 \text{ cm}$$

14-29. $f_r = 1.6\ GHz,\ \epsilon_r = 2.2,\ h = 0.1575\ cm$

Using (14-69a)

$$F = \frac{8.791}{1.6\sqrt{2.2}} = 3.704$$

Therefore using (14-69)

$$a = \frac{3.704}{\left\{1 + (0.0123)\left[\ln(36.944) + 1.7726\right]\right\}^{1/2}} = 3.587\ cm$$

14-30. Desired frequency : $f = 900\ MHz \rightarrow \lambda = \frac{1}{3}\ m = 0.333\ m$

$\epsilon_r = 10.2,\quad h = 0.127\ cm.\quad : TM^z_{210}$

a. Resonant frequency ;

From equation (14-63a), $(k_\rho)^2 + (k_z)^2 = \omega_r^2 \mu\epsilon$ ($\because k_\rho = \chi_{mn}/a,\ k_z = \frac{p\pi}{h}$)

$\therefore TM^z_{210} \rightarrow (m=2, n=1, p=0)$.

$$(\chi_{mn}/a)^2 = \omega_r^2 \mu\epsilon = (2\pi f_{rmno})^2 \cdot \mu\epsilon$$

$$f_{rmno} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\left(\frac{\chi_{mn}}{a}\right),\quad f_{r210} = \frac{1}{2\pi\sqrt{\mu_0\epsilon_0}\cdot\sqrt{\epsilon_r}}\left(\frac{3.0542}{a}\right)$$

$$f_{r210} = \frac{3.0542}{2\pi a\sqrt{\epsilon_r}}\cdot c \qquad (\because c = 3\times10^8\ m)$$

b. Neglect fringing. $a = \dfrac{3.0542\cdot c}{2\pi(f_{r210})\sqrt{\epsilon_r}} = \dfrac{3.0542}{2\pi\cdot 3\cdot\sqrt{10.2}} = 0.057\ m$

$a = 5.7\ cm$

c. $E_\theta \simeq -j\dfrac{k\,e^{-jkr}}{4\pi r}(L_\phi + \eta N_\theta)$

$E_\phi \simeq +j\dfrac{k\,e^{-jkr}}{4\pi r}(L_\theta - \eta N_\theta)$,

Using the equivalent principle the cavity can be modeled as a circular loop antenna of radius a with magnetic current for general mode.

$$I_m = 2V_0\cos(m\phi')$$

Similar procedure like problem 14-22, $\vec{L} = \iint_S \vec{I}_m \cdot e^{+jkr'\cos\psi}\,ds'$

$$L_\theta = \iint_S M_\phi \cdot \cos\theta \cdot \sin(\phi-\phi')\, e^{jkr'\cos\psi}\,ds' \qquad \leftarrow X = ka\sin\theta$$

$$= 2V_0 a\cos\theta \int_0^{2\pi} \cos n\phi' \sin(\phi-\phi')\, e^{+jX\cos(\phi-\phi')}\,d\phi'$$

(Continued)

14-30. (Cont'd)

$$L_\theta = 2V_0\cos\theta\int_0^{2\pi}\frac{1}{4j}\left(e^{jm\phi'}+e^{-jm\phi'}\right)\left(e^{j(\phi-\phi')}-e^{-j(\phi-\phi')}\right)e^{jX\cos(\phi-\phi')}\,d\phi'$$

$$= a\cdot\frac{V_0\cos\theta}{2j}\int_0^{2\pi}\left[e^{j(m\phi'+\phi-\phi')}-e^{jm\phi'-j(\phi-\phi')}+e^{-jm\phi'+j(\phi-\phi')}-e^{-j(m\phi'+\phi-\phi')}\right]\times e^{jX\cos(\phi-\phi')}\,d\phi'$$

$$= a\cdot\frac{V_0\cos\theta}{2j}\Bigg\{\int_0^{2\pi}\Big[\underbrace{e^{j(m-1)\phi'}\cdot e^{j\phi}}_{①}-\underbrace{e^{j(m+1)\phi'}\cdot e^{-j\phi}}_{②}+\underbrace{e^{-j(m+1)\phi'+j\phi}}_{③}-\underbrace{e^{-j(m-1)\phi'}e^{-j\phi}}_{④}\Big]e^{jX\cos(\phi-\phi')}\,d\phi'$$

① $$e^{j\phi}\int_0^{2\pi}e^{j(m-1)\phi'}\cdot e^{jX\cos(\phi-\phi')}\,d\phi' = e^{jm\phi}\int_0^{2\pi}e^{j(m-1)(\phi'-\phi)}\cdot e^{jX\cos(\phi-\phi')}\,d\phi'$$

$$= e^{jm\phi}\int_0^{2\pi}e^{j(1-m)(\phi-\phi')}e^{jX\cos(\phi-\phi')}\,d\phi' = e^{jm\phi}\cdot\frac{2\pi}{j^{-(1-m)}}J_{(1-m)}(X)$$

② $$e^{-j\phi}\int_0^{2\pi}e^{j(m+1)\phi'}e^{jX\cos(\phi-\phi')}\,d\phi' = e^{+jm\phi}\cdot\int_0^{2\pi}e^{j(m+1)(\phi'-\phi)}\cdot e^{jX\cos(\phi'-\phi)}\,d\phi'$$

$$= e^{jm\phi}\cdot\frac{2\pi}{j^{-(m+1)}}\cdot J_{m+1}(X)$$

③ $$e^{j\phi}\int_0^{2\pi}e^{-j(m+1)\phi'}e^{jX\cos(\phi-\phi')}\,d\phi' = e^{-jm\phi}\cdot\int_0^{2\pi}e^{j(m+1)(\phi-\phi')}e^{jX\cos(\phi-\phi')}\,d\phi'$$

$$= e^{-jm\phi}\cdot\frac{2\pi}{j^{-(m+1)}}\cdot J_{m+1}(X)$$

④ $$e^{-j\phi}\int_0^{2\pi}e^{-j(m-1)\phi'}\cdot e^{jX\cos(\phi-\phi')}\,d\phi' = e^{-jm\phi}\int e^{j(m-1)(\phi-\phi')}\cdot e^{jX\cos(\phi-\phi')}\,d\phi'$$

$$= e^{-jm\phi}\cdot\frac{2\pi}{j^{-(m-1)}}\cdot J_{m-1}(X)\quad,\ \text{used } J_m(X)=\frac{j^{-n}}{2\pi}\int_0^{2\pi}e^{jnu}e^{jX\cos u}\,du$$

$$\therefore L_\theta = a\pi\frac{V_0\cos\theta}{j}\Big[e^{jm\phi}(j)^{m-1}\cdot J_{m-1}(X)-e^{jm\phi}(j)^{m+1}J_{m+1}(X)+e^{-jm\phi}\cdot(j)^{m+1}\cdot J_{m+1}(X)$$

$$-e^{-jm\phi}\cdot(j)^{m-1}\cdot J_{m-1}(X)\Big]$$

$$= -j2\pi aV_0\cos\theta\left[\sin m\phi\cdot(j)^m J_{m-1}(X)+\sin m\phi\,(j)^m J_{m+1}(X)\right]$$

$$\therefore E_\phi = +j\frac{ke^{-jkr}}{4\pi r}\left[-j2\pi\cdot V_0\cos\theta\sin m\phi\,(J_{m-1}(X)+J_{m+1}(X))\right](j)^m$$

$$= (j)^m\cdot\frac{ke^{-jkr}}{2r}\left[V_0 a\cos\theta\cdot\sin m\phi\,(J_{m-1}(X)+J_{m+1}(X))\right]$$

$$L_\phi = 2V_0 a\int_0^{2\pi}\cos m\phi'\cdot\cos(\phi-\phi')\,e^{jka\sin\theta\cos(\phi-\phi')}\,d\phi'$$

(Continued)

14-30 (Cont'd)

$$L_\phi = 2V_0 a \int_0^{2\pi} \cos m\phi' \cdot \cos(\phi-\phi') e^{jka\sin\theta\cos(\phi-\phi')} d\phi'$$

$$X = ka\sin\theta .$$

$$L_\phi = 2V_0 a \int_0^{2\pi} \frac{1}{4}\left[e^{jm\phi'}+e^{-jm\phi'}\right]\left[e^{j(\phi-\phi')}+e^{-j(\phi-\phi')}\right] e^{jX\cos(\phi-\phi')} d\phi'$$

$$= \frac{V_0 a}{2}\int_0^{2\pi}\Big[\underbrace{e^{j(m-1)\phi'}\cdot e^{j\phi}}_{①} + \underbrace{e^{-j(m+1)\phi'}e^{j\phi}}_{②} + \underbrace{e^{j(m+1)\phi'}e^{-j\phi}}_{③} + \underbrace{e^{-j(m-1)\phi'}e^{-j\phi}}_{④}\Big] \times e^{jX\cos(\phi-\phi')}\, d\phi'$$

① : $\int_0^{2\pi} e^{j\phi}\cdot e^{j(m-1)\phi'} e^{jX\cos(\phi-\phi')} d\phi' = e^{jm\phi}\cdot\int_0^{2\pi} e^{j(m-1)(\phi'-\phi)} e^{jX\cos(\phi'-\phi)} d\phi'$

$= e^{jm\phi}\cdot \frac{2\pi}{j^{-(m-1)}} J_{m-1}(X)$

② $\int_0^{2\pi} e^{j\phi}\cdot e^{-j(m+1)\phi'} e^{jX\cos(\phi-\phi')} d\phi' = e^{-jm\phi}\int_0^{2\pi} e^{j(m+1)(\phi-\phi')}\cdot e^{jX\cos(\phi-\phi')} d\phi'$

$= e^{-jm\phi}\cdot \frac{2\pi}{j^{-(m+1)}}\cdot J_{m+1}(X)$

③ $\int_0^{2\pi} e^{-j\phi}\cdot e^{j(m+1)\phi'} e^{jX\cos(\phi-\phi')} d\phi' = e^{jm\phi}\cdot\int_0^{2\pi} e^{j(m+1)(\phi'-\phi)} e^{jX\cos(\phi'-\phi)} d\phi'$

$= e^{jm\phi}\cdot \frac{2\pi}{j^{-(m+1)}}\cdot J_{m+1}(X)$

④ $\int_0^{2\pi} e^{-j\phi}\cdot e^{-j(m-1)\phi'}\cdot e^{jX\cos(\phi-\phi')} d\phi' = e^{-jm\phi}\int_0^{2\pi} e^{j(m-1)(\phi-\phi')} e^{jX\cos(\phi-\phi')} d\phi'$

$= e^{-jm\phi}\cdot \frac{2\pi}{j^{-(m-1)}} J_{m-1}(X)$ .

$$\therefore L_\phi = \frac{V_0 a}{2}\left[2\pi\left\{\left(e^{jm\phi}+e^{-jm\phi}\right)(j)^{m-1}\cdot J_{m-1}(X) + \left(e^{jm\phi}+e^{-jm\phi}\right)(j)^{m+1}\cdot J_{m+1}(X)\right\}\right]$$

$$= V_0 a\cdot\pi\left[2\cos m\phi\cdot(j)^m(-j)J_{m-1}(X) + 2\cos m\phi\cdot(j)^m(j)J_{m+1}(X)\right]$$

$$\therefore L_\phi = 2\pi V_0 a\cos m\phi\left[(-j)(j)^m\right]\left[J_{m-1}(X) - J_{m+1}(X)\right]$$

(Continued)

14-30 (Cont'd), Assimilar procedure like $L_\theta$.

$$L_\phi = 2\pi V_0 a \cdot (-j)(j)^m \cdot \cos m\phi \left[ J_{m-1}(X) - J_{m+1}(X) \right],$$

$$\therefore E_\theta = -jk \frac{e^{-jkr}}{4\pi r} (2\pi V_0 a (-j)(j)^m \cdot \cos m\phi \left[ J_{m-1}(X) - J_{m+1}(X) \right].$$

$$= -(j)^m \cdot \frac{k e^{-jkr}}{2r} \left[ V_0 \cdot a \cdot \cos m\phi \cdot (J_{m-1}(X) - J_{m+1}(X)) \right],$$

For $m=2$, the far field is ($\leftarrow X = ka\sin\theta$.)

$$E_\theta = \frac{k e^{-jkr}}{2r} \left[ V_0 a \cos 2\phi \, (J_1(X) - J_3(X)) \right]$$

$$E_\phi = -\frac{k e^{-jkr}}{2r} \left[ V_0 a \cdot \cos\theta \cdot \sin 2\phi \, (J_1(X) + J_3(X)) \right]$$

d.

E and H Plane Field Pattern of Circular Patch (TM210 mode)

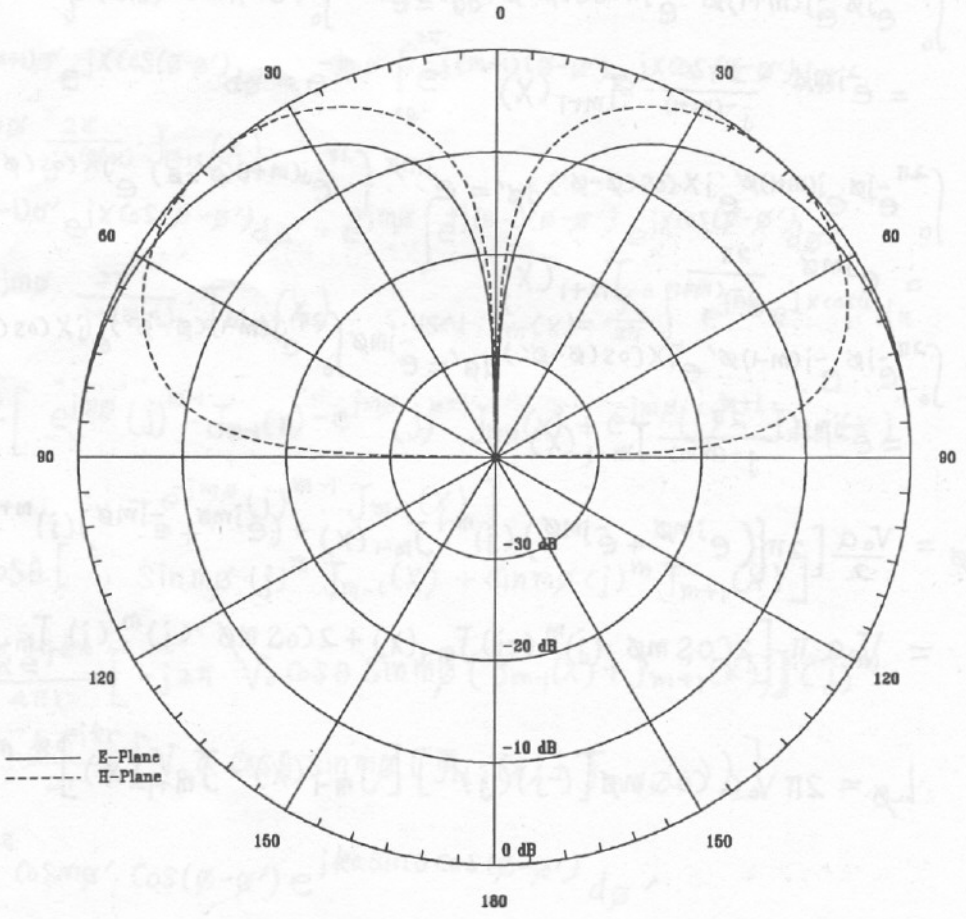



Fig. P14-30

(Continued)

14-30 (Cont'd) normalized azimuthal (x-y plane) amplitude pattern (dB)

e. $U = |E_\phi|^2 + |E_\theta|^2$, $|_{At\ \theta=90°}$. $U \propto (\cos 2\phi)^2$. At $\theta = 90°$

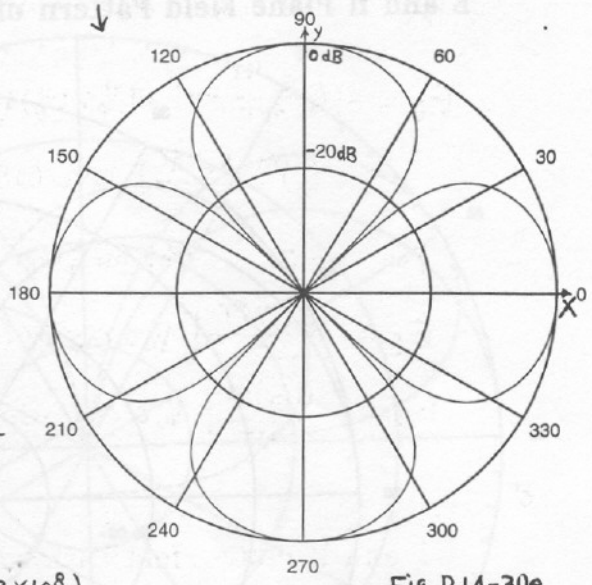



Fig P.14-30e

f. Directivity TM210 mode

$D_o = D_\theta + D_\phi$

$$D_\theta = \frac{4\pi U_{\theta\,max}}{(P_{rad})_\theta + (P_{rad})_\phi} = 4.3739$$

$$D_\phi = \frac{4\pi U_{\phi\,max}}{(P_{rad})_\theta + (P_{rad})_\phi} = 1.4866$$

$\therefore D_o = D_\theta + D_\phi = 4.3739 + 1.4866 = 5.8605$

$D_o = 7.6\,dB$

14-31 a. $(f_r)_{310} = \frac{4.2012}{2\pi a\sqrt{\epsilon_r}} \cdot C$ $(\leftarrow C = 3\times10^8 m)$

b. $a = \frac{4.2012 \cdot C}{2\pi (f_r)_{310}\sqrt{\epsilon_r}} = \frac{4.2012}{2\pi \cdot 3}\frac{1}{\sqrt{10.2}} = 0.0698\,m = 6.98\,cm$

c. far-zone field. from part (c) in problem 14-30

$E_\theta = (-j)^3 \frac{ke^{jkr}}{2r}[V_o \cdot a \cdot \cos 3\phi \cdot (J_2(X) - J_4(X))]$, $X = ka\sin\theta$

$= j\frac{ke^{jkr}}{2r}[V_o \cdot a \cdot \cos 3\phi\,(J_2(ka\sin\theta) - J_4(ka\sin\theta))]$

$E_\phi = -j\frac{ke^{-jkr}}{2r}[V_o \cdot a \cdot \sin 3\phi \cdot \cos\theta \cdot [J_2(ka\sin\theta) + J_4(ka\sin\theta)]]$

d. See the Fig P14-31.

e. See the Right. ⇒

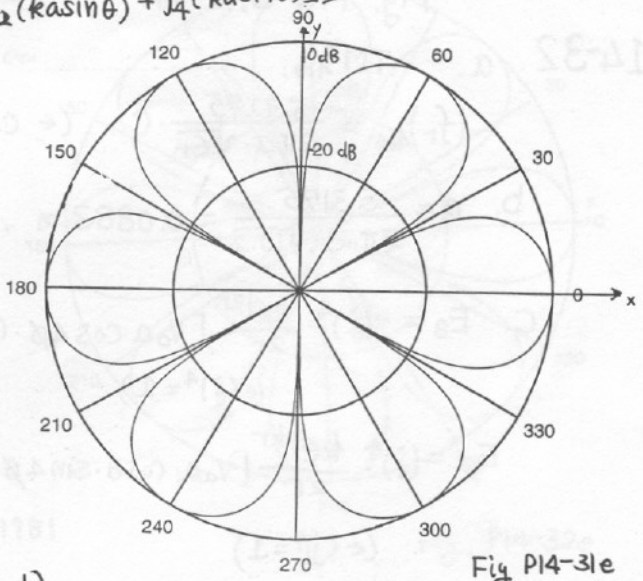



Fig P14-31e

f. $D_o = D_\theta + D_\phi$

$D_\theta = 5.98148$

$D_\phi = 1.111$

$D_o = 7.092 = 8.5\,dB$

(Continued)

14-31 (Cont'd)

E and H Plane Field Pattern of Circular Patch (TM310 mode)

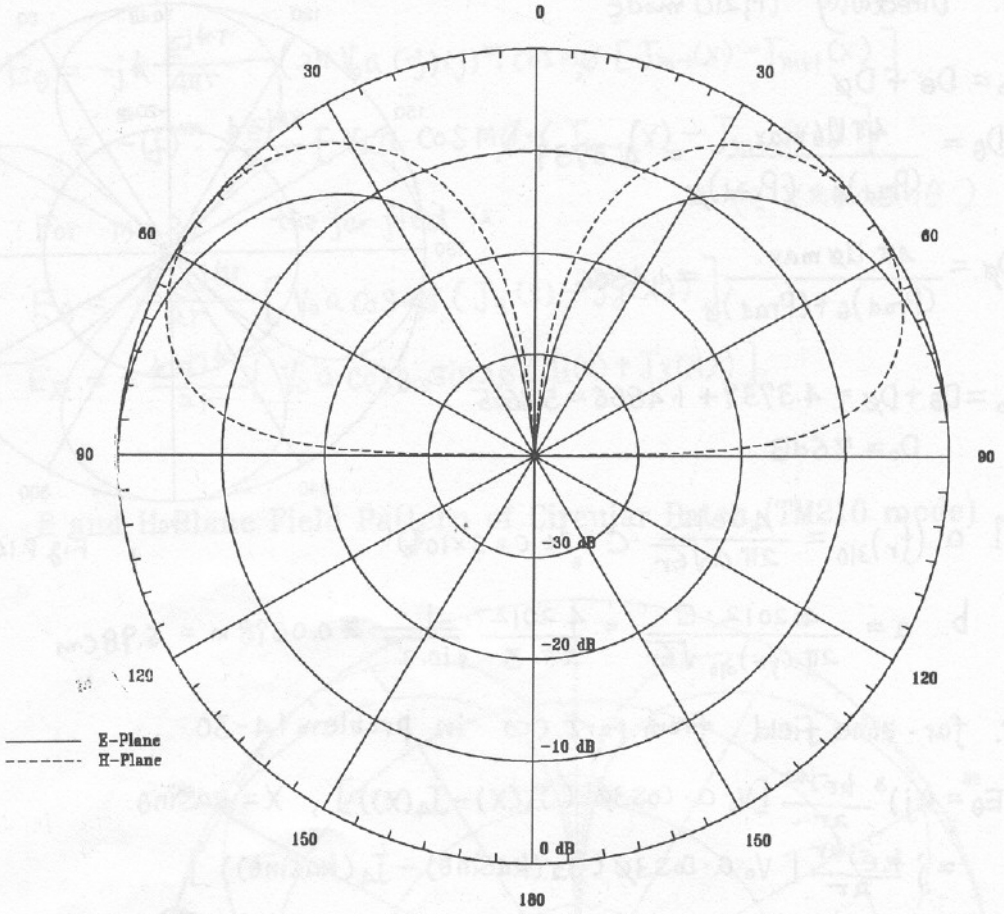



Fig. P14-31.

14-32. a. $TM^z_{410}$.

$$(f_r)_{410} = \frac{5.3175}{2\pi a \cdot \sqrt{\epsilon_r}} \cdot c \qquad (\leftarrow c = 3\times10^8 \text{ m})$$

b. $$a = \frac{5.3175}{2\pi \cdot 3 \cdot \sqrt{10.2}} = 0.0883 \text{ m}, \quad a = 8.83 \text{ cm}$$

c. $$E_\theta = -(j)^4 \cdot \frac{ke^{jkr}}{2r}\left[V_0 a \cdot \cos 4\phi \cdot (J_3(k a \sin\theta) - J_5(k a \sin\theta))\right]$$

$(\leftarrow (j)^4 = 1)$

$$E_\phi = (j)^4 \cdot \frac{ke^{jkr}}{2r}\left[V_0 a \cdot \cos\theta \cdot \sin 4\phi \cdot (J_3(k a \sin\theta) + J_5(k a \sin\theta))\right]$$

$(\leftarrow (j)^4 = 1)$

(Continued)

14-32. (Cont'd)

d. E and H Plane Field Pattern of Circular Patch (TM410 mode)

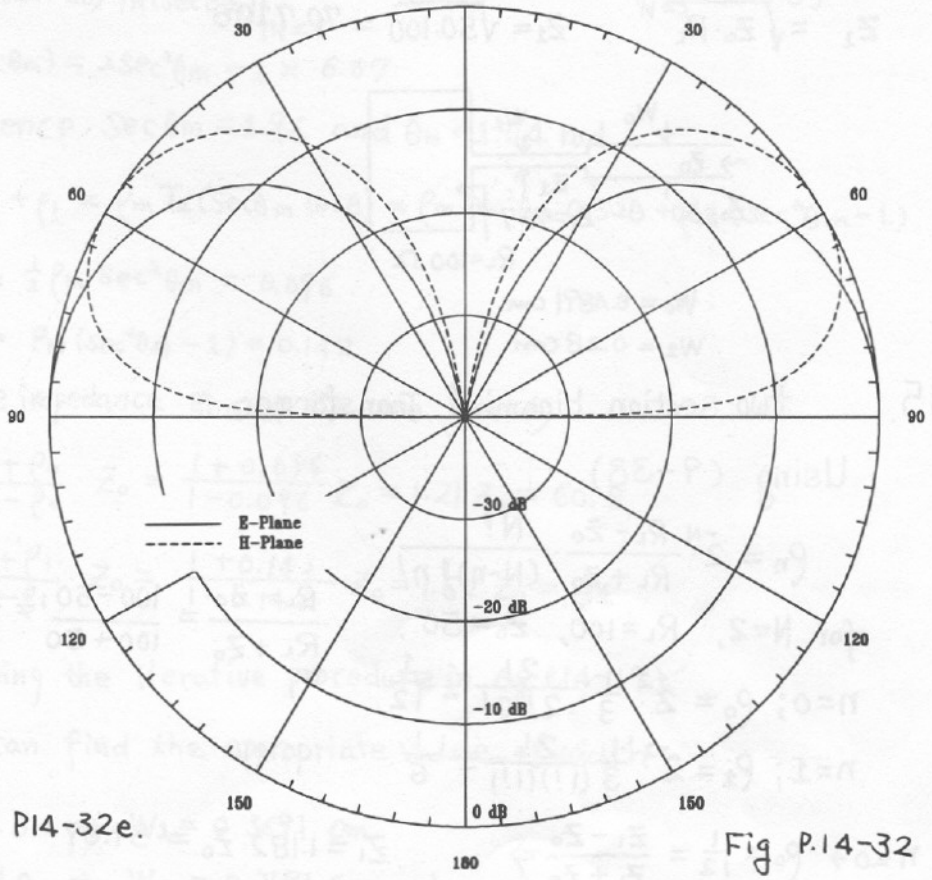



Fig P.14-32

e. See Fig. P14-32e.

f. $D_o = D_\theta + D_\phi = 7.0929 + 0.853 = 7.945$
$= 9\,dB$

14-33. $f = 10\,GHz, \Rightarrow \lambda = 3\,cm$, $d = 0.1\,cm$
$\epsilon_r = .2$, $h = 0.1575\,cm$

$$jX_p = j\frac{Z_o}{\sqrt{\epsilon_r}} \cdot \tan(\sqrt{\epsilon_r} \cdot k_o h)$$

$$= j\frac{120 \cdot \pi}{\sqrt{2.2}} \tan\left(\sqrt{2.2}\ \frac{2\pi}{3} \cdot 0.1575\right)$$

$$= j\,254.167(0.53245) = j135.33$$

From K.R Carver & J.W Mink.
"Microstrip Antenna technology"
IEEE Trans. AP Vol 29. pp2-24. 1981

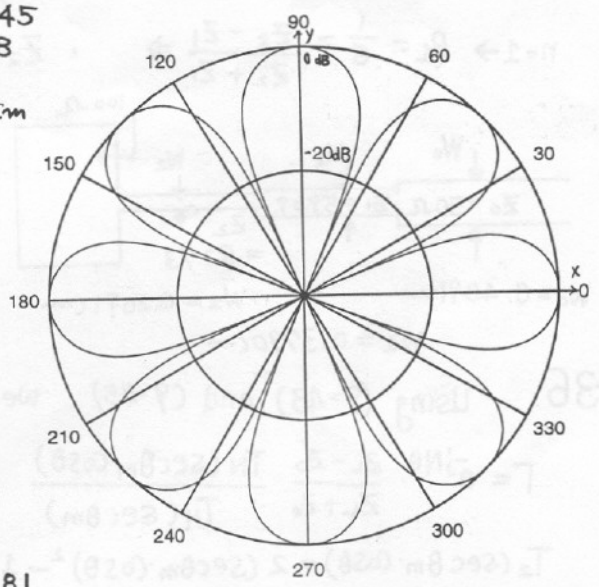
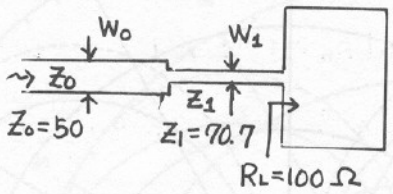



Fig. P14-32e.

14-34. $\frac{\lambda}{4}$ impedance transformer, $Z_0 = 50$ ohms, $\epsilon_r = 2.2$

$R_L = 100$ ohms,

$Z_1 = \sqrt{Z_0 \cdot R_L}$ $\quad \therefore Z_1 = \sqrt{50 \cdot 100} = 70.7106$




$\therefore W_0 = 0.4891$ cm

$W_1 = 0.28$ cm

14-35. two section binomial Transformer.

Using (9-38)

$$\rho_n = 2^{-N} \frac{R_L - Z_0}{R_L + Z_0} \cdot \frac{N!}{(N-n)!\,n!}$$

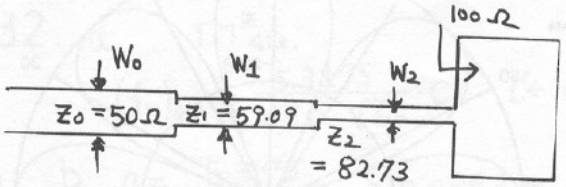
for $N=2$, $R_L = 100$, $Z_0 = 50$, $\quad \frac{R_L - Z_0}{R_L + Z_0} = \frac{100-50}{100+50} = \frac{1}{3}$

$n=0$; $\rho_0 = 2^{-2} \cdot \frac{1}{3} \cdot \frac{2!}{2! \cdot 0!} = \frac{1}{12}$.

$n=1$; $\rho_1 = 2^{-2} \cdot \frac{1}{3} \cdot \frac{2!}{(1!)(1!)} = \frac{1}{6}$

$n=0 \rightarrow \rho_0 = \frac{1}{12} = \frac{Z_1 - Z_0}{Z_1 + Z_0} \Rightarrow$ , $Z_1 = 1.182 \cdot Z_0 = 59.09$

$n=1 \rightarrow \rho_1 = \frac{1}{6} = \frac{Z_2 - Z_1}{Z_2 + Z_1} \Rightarrow$ , $Z_2 = 1.399\ Z_1 = 82.73$




$W_0 = 0.4891$cm $\qquad W_2 = 0.2091$cm

$W_1 = 0.3790$cm

14-36. Using (9-43) and (9-45), we obtain

$$\Gamma = e^{-jN\theta} \cdot \frac{Z_L - Z_0}{Z_L + Z_0} \cdot \frac{T_N(\sec\theta_m \cos\theta)}{T_N(\sec\theta_m)} \quad \cdots$$

$T_2(\sec\theta_m \cdot \cos\theta) = 2(\sec\theta_m \cdot \cos\theta)^2 - 1 = 2\sec^2\theta_m(1+\cos 2\theta) - 1$

(Continued)

14-36(Cont'd) Let the maximum tolerable value of $\rho$ be $\rho_m = 0.05$.

$$\rho_m = \left.\frac{Z_L - Z_0}{(Z_L + Z_0)\, T_N(\sec\theta_m)}\right|_{N=2} = 0.05 \quad , \quad \therefore T_N(\sec\theta_m)\Big|_{N=2} = \frac{1}{3}\cdot\frac{1}{0.05} = 6.67$$

$$T_2(\sec\theta_m) = 2\sec^2\theta_m - 1 = 6.67$$

and hence $\sec\theta_m = 1.96$ and $\theta_m = 1.04$ rad.

$$2\rho_0 \cos 2\theta + \rho_1 = \rho_m T_2(\sec\theta_m \cos\theta) = \rho_m \sec^2\theta_m \cdot \cos 2\theta + \rho_m(\sec^2\theta_m - 1)$$

$$\therefore \quad \rho_0 = \tfrac{1}{2}\rho_m \sec^2\theta_m = 0.096$$

$$\rho_1 = \rho_m(\sec^2\theta_m - 1) = 0.142$$

Thus, the impedance $Z_1$ and $Z_2$ are given by

$$Z_1 = \frac{1+\rho_0}{1-\rho_0}\cdot Z_0 = \frac{1+0.096}{1-0.096} Z_0 = 1.21\, Z_0 = 60.5$$

$$Z_2 = \frac{1+\rho_1}{1-\rho_1}\cdot Z_0 = \frac{1+0.142}{1-0.142} Z_0 = 1.62\, Z_0 = 81.$$

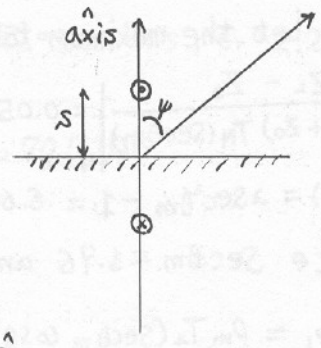
By applying the iterative procedure in eq (14-19)

we can find the appropriate value of width.

$Z_1 = 60.5 \Rightarrow W_1 = 0.3691$ cm

$Z_2 = 81.0 \Rightarrow W_2 = 0.2181$ cm

15-1.




Array Factor

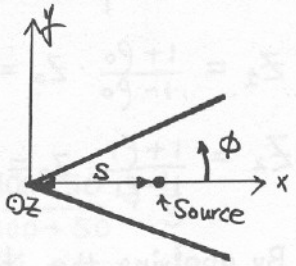
$= C\left[e^{jks\cos\psi_1} - e^{-jks\cos\psi_2}\right]$

(in far-zone: $\psi_1 = \psi_2 = \psi$, )

$AF = 2j\sin(ks\cos\psi)$

$\leftarrow \cos\psi = \hat{axis}\cdot\hat{a}_r$

15-2. a. For all three corner reflectors ($\alpha = 60°, 45°, 30°$), the geometrical coordinate system shown in the adjacent figure will be used. The sources will be numbered so that the feed will be #1. The images will be designated as #2, #3, ... (in a counter clock wise rotation) as shown in Figure 15.4(b) for the 90° corner reflector.




$\alpha = 60°$

Using the source arrangement of Figure 15.3(b)

$$E(r,\theta,\phi) = \left(\frac{e^{-jkr_1}}{r_1} + \frac{e^{-jkr_2}}{r_2} + \frac{e^{-jkr_3}}{r_3} + \frac{e^{-jkr_4}}{r_4} + \frac{e^{-jkr_5}}{r_5} + \frac{e^{-jkr_6}}{r_6}\right) f(\theta,\phi)$$

for the main source (#1) and the five images. For far-field observations

For phase terms

$r_1 \simeq r - s\cos\psi_1 = r - s(\hat{a}_x\cdot\hat{a}_r) = r - s\sin\theta\cos\phi$

$r_2 \simeq r - s\cos\psi_2 = r - s[(0.5\hat{a}_x + 0.866\hat{a}_y)\cdot\hat{a}_r] = r - s(0.5\sin\theta\cos\phi + 0.866\sin\theta\sin\phi)$

$r_3 \simeq r - s\cos\psi_3 = r - s[(-0.5\hat{a}_x + 0.866\hat{a}_y)\cdot\hat{a}_r] = r - s(-0.5\sin\theta\cos\phi + 0.866\sin\theta\sin\phi)$

$r_4 \simeq r - s\cos\psi_4 = r - s(-\hat{a}_x\cdot\hat{a}_r) = r + s\sin\theta\cos\phi$

$r_5 \simeq r - s\cos\psi_5 = r - s[(-0.5\hat{a}_x - 0.866\hat{a}_y)\cdot\hat{a}_r] = r + s(0.5\sin\theta\cos\phi + 0.866\sin\theta\sin\phi)$

$r_6 \simeq r - s\cos\psi_6 = r - s[(0.5\hat{a}_x - 0.866\hat{a}_y)\cdot\hat{a}_r] = r + s(-0.5\sin\theta\cos\phi + 0.866\sin\theta\sin\phi)$

where $\hat{a}_r = \hat{a}_x\sin\theta\cos\phi + \hat{a}_y\sin\theta\sin\phi + \hat{a}_z\cos\theta$

(Continued)

15-2 (Cont'd)

For amplitude terms

$$r_1 \simeq r_2 \simeq r_3 \simeq r_4 \simeq r_5 \simeq r_6 \simeq r$$

Making these substitutions and combining terms (first with fourth, second with fifth, and third with sixth), we can write that

$$E = f(\theta,\phi)\frac{e^{-jkr}}{r} 2\{\sin(ks\sin\theta\cos\phi) - \sin[ks(0.5\sin\theta\cos\phi + 0.866\sin\theta\sin\phi)] + \sin[ks(-0.5\sin\theta\cos\phi + 0.866\sin\theta\sin\phi)]\}$$

Using the identities of

$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$ reduces the field to

$$\frac{E}{E_0} = 2\left\{\sin(ks\sin\theta\cos\phi) - 2\sin\left(\frac{ks}{2}\sin\theta\cos\phi\right)\cos\left(\frac{\sqrt{3}}{2}ks\sin\theta\sin\phi\right)\right\}$$

where $E_0 = f(\theta,\phi)\frac{e^{-jkr}}{r}$

Utilizing the identity of $\sin 2x = 2\sin x\cos x$

we can write in the final form that

$$\frac{E}{E_0} = 4\sin\left(\frac{ks}{2}\sin\theta\cos\phi\right)\left[\cos\left(\frac{ks}{2}\sin\theta\cos\phi\right) - \cos\left(\frac{\sqrt{3}}{2}ks\sin\theta\sin\phi\right)\right]$$

$\alpha = 45°$

Using the source arrangement of Figure 15.3(c)

$$E = \left(\frac{e^{-jkr_1}}{r_1} + \frac{e^{-jkr_2}}{r_2} + \frac{e^{-jkr_3}}{r_3} + \frac{e^{-jkr_4}}{r_4} + \frac{e^{-jkr_5}}{r_5} + \frac{e^{-jkr_6}}{r_6} + \frac{e^{-jkr_7}}{r_7} + \frac{e^{-jkr_8}}{r_8}\right) f(\theta,\phi)$$

For far-field observations

For phase terms

$$r_1 \simeq r - s\cos\psi_1 = r - s(\hat{a}_x \cdot \hat{a}_r) = r - s\sin\theta\cos\phi$$

$$r_2 = r - s\cos\psi_2 = r - s\left[\frac{1}{\sqrt{2}}(\hat{a}_x + \hat{a}_y)\cdot\hat{a}_r\right] = r - \frac{s}{\sqrt{2}}(\sin\theta\cos\phi + \sin\theta\sin\phi)$$

$$r_3 = r - s\cos\psi_3 = r - s(\hat{a}_y \cdot \hat{a}_r) = r - s\sin\theta\sin\phi$$

$$r_4 = r - s\cos\psi_4 = r - s\left[\frac{1}{\sqrt{2}}(-\hat{a}_x + \hat{a}_y)\cdot\hat{a}_r\right] = r + \frac{s}{\sqrt{2}}(\sin\theta\cos\phi - \sin\theta\sin\phi)$$

$$r_5 = r - s\cos\psi_5 = r - s(-\hat{a}_x \cdot \hat{a}_r) = r + s\sin\theta\cos\phi$$

$$r_6 = r - s\cos\psi_6 = r - s\left[\frac{1}{\sqrt{2}}(-\hat{a}_x - \hat{a}_y)\cdot\hat{a}_r\right] = r + \frac{s}{\sqrt{2}}(\sin\theta\cos\phi + \sin\theta\sin\phi)$$

$$r_7 = r - s\cos\psi_7 = r - s(-\hat{a}_y \cdot \hat{a}_r) = r + s\sin\theta\sin\phi$$

$$r_8 = r - s\cos\psi_8 = r - s\left[\frac{1}{\sqrt{2}}(\hat{a}_x - \hat{a}_y)\cdot\hat{a}_r\right] = r - \frac{s}{\sqrt{2}}(\sin\theta\cos\phi - \sin\theta\sin\phi)$$

For amplitude terms

$$r_1 \simeq r_2 \simeq r_3 \simeq r_4 \simeq r_5 \simeq r_6 \simeq r_7 \simeq r_8 \simeq r$$

Making these substitutions and combining terms (first with fifth, second with sixth, third with seventh, fourth with eighth), we can write

(Continued)

15-2 (Cont'd)

$$\frac{E}{E_0} = 2\left\{\cos(ks\sin\theta\cos\phi) + \cos(ks\sin\theta\sin\phi) - 2\cos\left(\frac{ks}{\sqrt{2}}\sin\theta\cos\phi\right)\cos\left(\frac{ks}{2}\sin\theta\sin\phi\right)\right\}$$

where $E_0 = f(\theta,\phi)\frac{e^{-jkr}}{r}$

$\alpha = 30°$

The procedure for this reflector follows those of the others.

Using the geometry of Figure 15.3(d)

$$E = \left(\frac{e^{-jkr_1}}{r_1} + \frac{e^{-jkr_2}}{r_2} + \frac{e^{-jkr_3}}{r_3} + \frac{e^{-jkr_4}}{r_4} + \frac{e^{-jkr_5}}{r_5} + \frac{e^{-jkr_6}}{r_6} + \frac{e^{-jkr_7}}{r_7} + \frac{e^{-jkr_8}}{r_8}\right.$$
$$\left. + \frac{e^{-jkr_9}}{r_9} + \frac{e^{-jkr_{10}}}{r_{10}} + \frac{e^{-jkr_{11}}}{r_{11}} + \frac{e^{-jkr_{12}}}{r_{12}}\right) f(\theta,\phi)$$

For far field observations

For phase terms

$r_1 = r - s\cos\psi_1 = r - s(\hat{a}_x\cdot\hat{a}_r) = r - s\sin\theta\cos\phi$

$r_2 = r - s\cos\psi_2 = r - s\left[\frac{1}{2}(\sqrt{3}\hat{a}_x + \hat{a}_y)\cdot\hat{a}_r\right] = r - \frac{s}{2}(\sqrt{3}\sin\theta\cos\phi + \sin\theta\sin\phi)$

$r_3 = r - s\cos\psi_3 = r - s\left[\frac{1}{2}(\hat{a}_x + \sqrt{3}\hat{a}_y)\cdot\hat{a}_r\right] = r - \frac{s}{2}(\sin\theta\cos\phi + \sqrt{3}\sin\theta\sin\phi)$

$r_4 = r - s\cos\psi_4 = r - s[\hat{a}_y\cdot\hat{a}_r] = r - s\sin\theta\sin\phi$

$r_5 = r - s\cos\psi_5 = r - s\left[\frac{1}{2}(-\hat{a}_x + \sqrt{3}\hat{a}_y)\cdot\hat{a}_r\right] = r - \frac{s}{2}(-\sin\theta\cos\phi + \sqrt{3}\sin\theta\sin\phi)$

$r_6 = r - s\cos\psi_6 = r - s\left[\frac{1}{2}(-\sqrt{3}\hat{a}_x + \hat{a}_y)\cdot\hat{a}_r\right] = r - \frac{s}{2}(-\sqrt{3}\sin\theta\cos\phi + \sin\theta\sin\phi)$

$r_7 = r - s\cos\psi_7 = r - s(-\hat{a}_x\cdot\hat{a}_r) = r + s\sin\theta\cos\phi$

$r_8 = r - s\cos\psi_8 = r - s\left[\frac{1}{2}(-\sqrt{3}\hat{a}_x - \hat{a}_y)\cdot\hat{a}_r\right] = r + \frac{s}{2}(\sqrt{3}\sin\theta\cos\phi + \sin\theta\sin\phi)$

$r_9 = r - s\cos\psi_9 = r - s\left[\frac{1}{2}(-\hat{a}_x - \sqrt{3}\hat{a}_y)\cdot\hat{a}_r\right] = r + \frac{s}{2}(\sin\theta\cos\phi + \sqrt{3}\sin\theta\sin\phi)$

$r_{10} = r - s\cos\psi_{10} = r - s(-\hat{a}_y\cdot\hat{a}_r) = r + \sin\theta\sin\phi$

$r_{11} = r - s\cos\psi_{11} = r - s\left[\frac{1}{2}(\hat{a}_x - \sqrt{3}\hat{a}_y)\cdot\hat{a}_r\right] = r + \frac{s}{2}(-\sin\theta\cos\phi + \sqrt{3}\sin\theta\sin\phi)$

$r_{12} = r - s\cos\psi_{12} = r - s\left[\frac{1}{2}(\sqrt{3}\hat{a}_x - \hat{a}_y)\cdot\hat{a}_r\right] = r + \frac{s}{2}(-\sqrt{3}\sin\theta\cos\phi + \sin\theta\sin\phi)$

For amplitude terms

Making these substitutions and combining terms (first with seventh, second with eighth, third with ninth, fourth with tenth, fifth with eleventh sixth with twelfth), we can write

$$\frac{E}{E_0} = 2\left\{\cos(ks\sin\theta\cos\phi) - 2\cos\left(\frac{\sqrt{3}}{2}ks\sin\theta\cos\phi\right)\cos\left(\frac{1}{2}ks\sin\theta\sin\phi\right)\right.$$
$$\left. - \cos(ks\sin\theta\sin\phi) + 2\cos\left(\frac{1}{2}ks\sin\theta\cos\phi\right)\cos\left(\frac{\sqrt{3}}{2}\cdot ks\sin\theta\sin\phi\right)\right\}$$

where $E_0 = f(\theta,\phi)\frac{e^{-jkr}}{r}$

(Continued)

15-2. (Cont'd)

b.

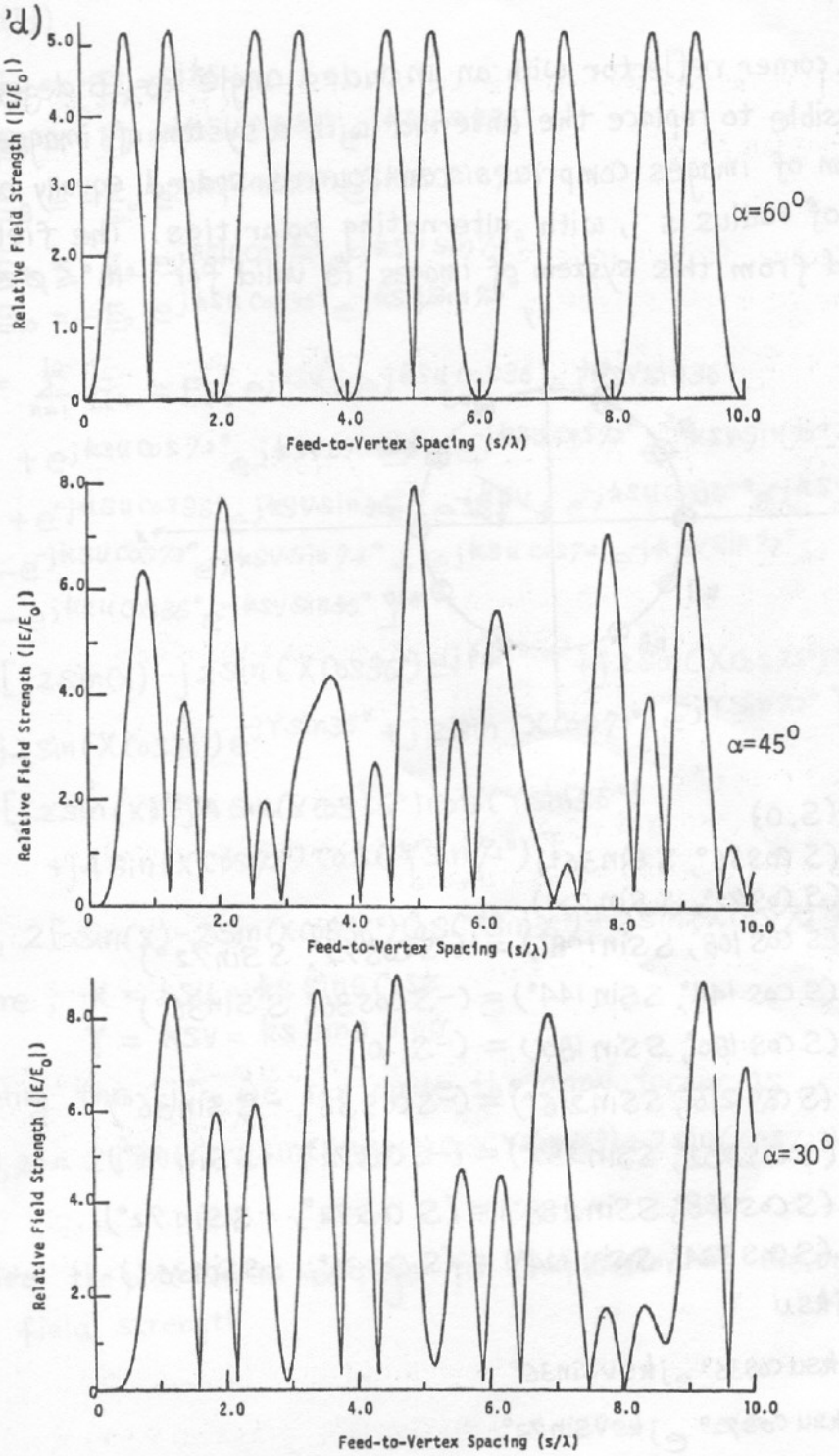
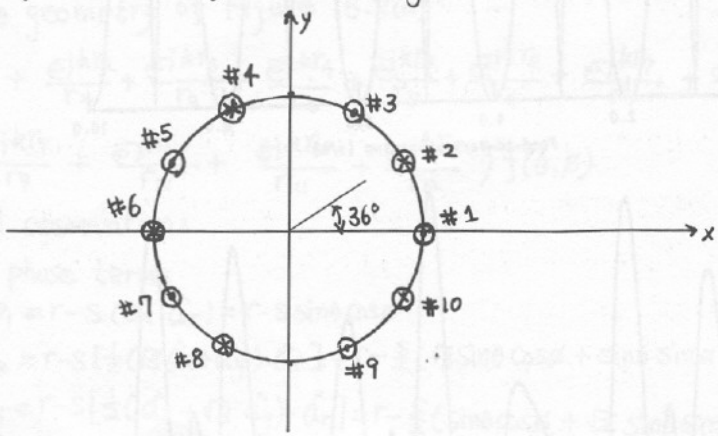



Fig. P15-2. Relative field strengths along the axis ($\theta=90^o$, $\phi=0^o$) for $\alpha=60^o$, $45^o$, $30^o$ corner reflectors as a function of feed-to-vertex spacing.

15-3.

a. For a corner reflector with an included angle $\alpha = 36$ degrees, it is possible to replace the antenna with a system of images. The system of images comprises ten sources spaced equally about a circle of radius s, with alternating polarities. The field determined from this system of images is valid for $-18° \le \phi \le 18°$.




$(x_1, y_1) = (S, 0)$

$(x_2, y_2) = (S\cos 36°, S\sin 36°)$

$(x_3, y_3) = (S\cos 72°, S\sin 72°)$

$(x_4, y_4) = (S\cos 108°, S\sin 108°) = (-S\cos 72°, S\sin 72°)$

$(x_5, y_5) = (S\cos 144°, S\sin 144°) = (-S\cos 36°, S\sin 36°)$

$(x_6, y_6) = (S\cos 180°, S\sin 180°) = (-S, 0)$

$(x_7, y_7) = (S\cos 216°, S\sin 216°) = (-S\cos 36°, -S\sin 36°)$

$(x_8, y_8) = (S\cos 252°, S\sin 252°) = (-S\cos 72°, -S\sin 72°)$

$(x_9, y_9) = (S\cos 288°, S\sin 288°) = (S\cos 72°, -S\sin 72°)$

$(x_{10}, y_{10}) = (S\cos 324°, S\sin 324°) = (S\cos 36°, -S\sin 36°)$

$\underline{E}_1 = \underline{E}_0 e^{jksu}$

$\underline{E}_2 = -\underline{E}_0 e^{jksu\cos 36°} e^{jksv\sin 36°}$

$\underline{E}_3 = \underline{E}_0 e^{jksu\cos 72°} e^{jksv\sin 72°}$

$\underline{E}_4 = -\underline{E}_0 e^{-jksu\cos 72°} e^{jksv\sin 72°}$

$\underline{E}_5 = \underline{E}_0 e^{-jksu\cos 36°} e^{jksv\sin 36°}$

(Continued)

15-3 (Cont'd)

$$\underline{E}_6 = -\underline{E}_0 e^{-jksu}$$

$$\underline{E}_7 = \underline{E}_0 e^{jksu\cos 36^\circ} e^{-jksv\sin 36^\circ}$$

$$\underline{E}_8 = -\underline{E}_0 e^{-jksu\cos 72^\circ} e^{-jksv\sin 72^\circ}$$

$$\underline{E}_9 = \underline{E}_0 e^{jksu\cos 72^\circ} e^{-jksv\sin 72^\circ}$$

$$\underline{E}_{10} = -\underline{E}_0 e^{jksu\cos 36^\circ} e^{-jksv\sin 72^\circ}$$

$$\underline{E} = \sum_{n=1}^{10} \underline{E}_n = \underline{E}_0 \left[ e^{jksu} - e^{jksu\cos 36^\circ} e^{jksv\sin 36^\circ} \right.$$
$$+ e^{jksu\cos 72^\circ} e^{jksv\sin 72^\circ} - e^{-jksu\cos 72^\circ} e^{jksv\sin 72^\circ}$$
$$+ e^{-jksu\cos 36^\circ} e^{jksv\sin 36^\circ} - e^{-jksu} + e^{-jksu\cos 36^\circ} e^{-jksv\sin 36^\circ}$$
$$- e^{-jksu\cos 72^\circ} e^{-jksv\sin 72^\circ} + e^{jksu\cos 72^\circ} e^{-jksv\sin 72^\circ}$$
$$\left. - e^{jksu\cos 36^\circ} e^{-jksv\sin 36^\circ} \right]$$

$$= \underline{E}_0 \left[ j2\sin(X) - j2\sin(X\cos 36^\circ) e^{jY\sin 36^\circ} + j2\sin(X\cos 72^\circ) e^{jY\sin 72^\circ} \right.$$
$$\left. - j2\sin(X\cos 36^\circ) e^{-jY\sin 36^\circ} + j2\sin(X\cos 72^\circ) e^{-jY\sin 72^\circ} \right]$$

$$= \underline{E}_0 \left[ j2\sin(X) - j4\sin(X\cos 36^\circ)\cos(Y\sin 36^\circ) \right.$$
$$\left. + j4\sin(X\cos 72^\circ)\cos(Y\sin 72^\circ) \right]$$

$$= j\underline{E}_0 \cdot 2\left[ \sin(X) - 2\sin(X\cos 36^\circ)\cos(Y\sin 36^\circ) + 2\sin(X\cos 72^\circ)\cos(Y\sin 72^\circ) \right]$$

where; $X = ksu = ks\sin\theta\cos\phi$

$Y = ksv = ks\sin\theta\sin\phi$

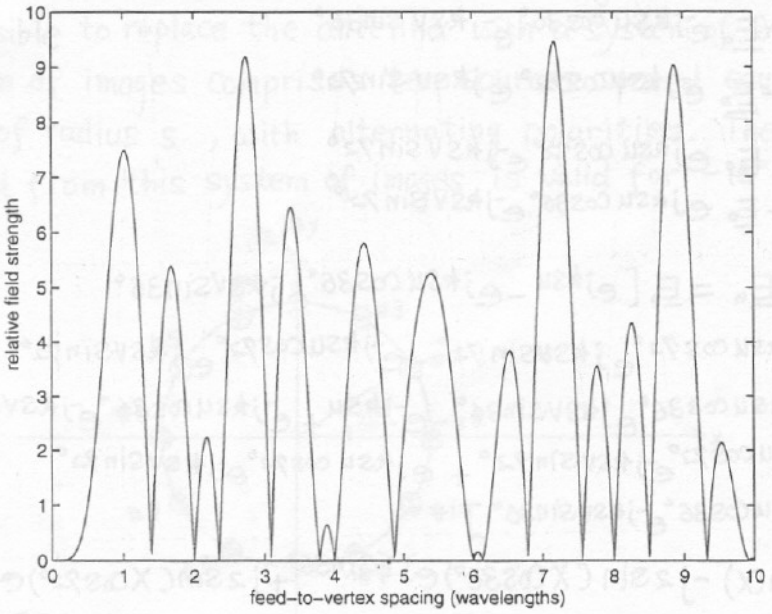
neglecting the "j", we can write the array factor as

$$AF(\theta,\phi) = 2\left[ \sin(X) - 2\sin(X\cos 36^\circ)\cos(Y\sin 36^\circ) + 2\sin(X\cos 72^\circ)\cos(Y\sin 72^\circ) \right]$$

b. See the plot on the next page for computation of the relative field strength.

(Continued)

15-3 (Cont'd)




C. $AF(\theta=90°, \phi=0°) = 2[\sin(ks) - 2\sin(ks\cdot\cos 36°) + 2\sin(ks\cos 72°)]$

$$\frac{d\,AF}{d(ks)} = 2[\cos(ks) - 2\cos 36°\cos(ks\cos 36°) + 2\cos 72°\cdot\cos(ks\cos 72°)]$$

The first zero of $\frac{d(AF)}{d(ks)}$ corresponding to a maximum in the relative field strength occurs for

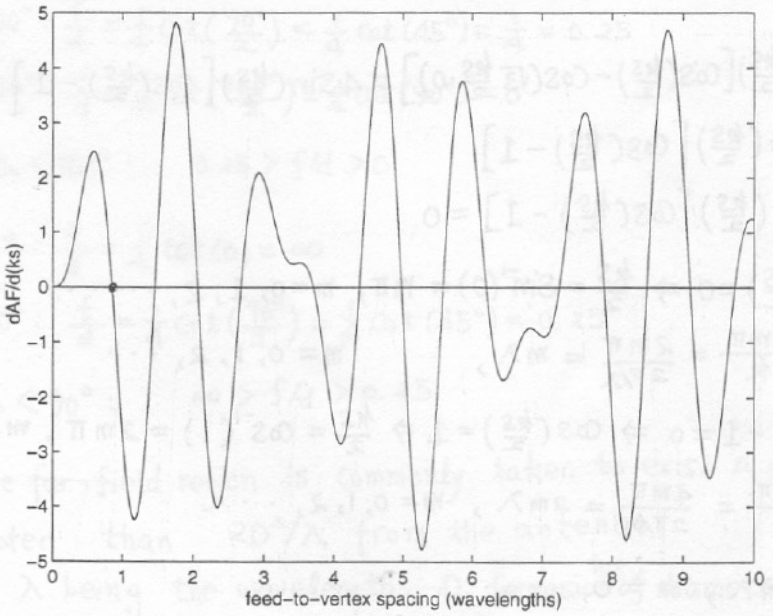
$ks = 6.415$ or $S = 1.021\lambda$

The relative field strength is given by

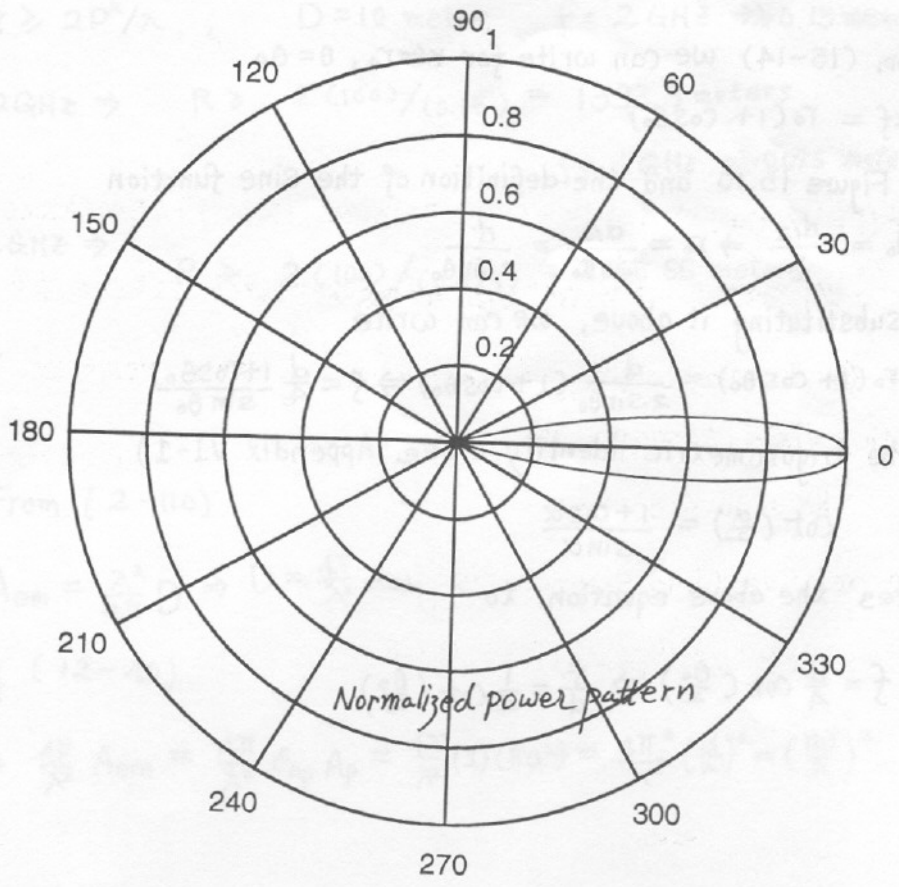
$$|AF(\theta=\tfrac{\pi}{2}, \phi=0)| = 7.482$$

(Continued)

15-3 (Cont'd)


dAF/d(ks)
feed-to-vertex spacing (wavelengths)

d.


90
120
60
150
30
180
0
210
330
240
300
270
0.8
0.6
0.4
0.2
Normalized power pattern

15-4. $AF = 4\sin\left(\frac{ks}{2}\sin\theta\cos\phi\right)\left[\cos\left(\frac{ks}{2}\sin\theta\cos\phi\right) - \cos\left(\sqrt{3}\cdot\frac{ks}{2}\cdot\sin\theta\sin\phi\right)\right]\Big|_{\substack{\theta=\frac{\pi}{2}\\ \phi=0^\circ}}$

$AF = 4\sin\left(\frac{ks}{2}\right)\left[\cos\left(\frac{ks}{2}\right) - \cos\left(\sqrt{3}\,\frac{ks}{2}\cdot 0\right)\right] = 4\sin\left(\frac{ks}{2}\right)\left[\cos\left(\frac{ks}{2}\right) - 1\right]$

$AF = 4\sin\left(\frac{ks}{2}\right)\left[\cos\left(\frac{ks}{2}\right) - 1\right]$

$AF = 4\sin\left(\frac{ks}{2}\right)\left[\cos\left(\frac{ks}{2}\right) - 1\right] = 0$

$\sin\left(\frac{ks}{2}\right) = 0 \Rightarrow \frac{ks}{2} = \sin^{-1}(0) = m\pi,\ m = 0, 1, 2, \cdots\cdots$

$s = \frac{2m\pi}{k} = \frac{2m\pi}{2\pi/\lambda} = m\lambda, \qquad m = 0, 1, 2, \cdots$

$\cos\left(\frac{ks}{2}\right) - 1 = 0 \Rightarrow \cos\left(\frac{ks}{2}\right) = 1 \Rightarrow \frac{ks}{2} = \cos^{-1}(1) = 2m\pi,\ m = 0, 1, \cdots$

$s = \frac{4m\pi}{k} = \frac{4m\pi}{2\pi/\lambda} = 2m\lambda,\ m = 0, 1, 2, \cdots$

$\therefore\ s = m\lambda,\ m = 0, 1, 2, \cdots$

15-5. From (15-14) we can write for $r' = r_0$, $\theta = \theta_0$

$2f = r_0(1 + \cos\theta_0)$

Using Figure 15.10 and the definition of the sine function

$\sin\theta_0 = \frac{d/2}{r_0} \Rightarrow r_0 = \frac{d/2}{\sin\theta_0} = \frac{d}{2\sin\theta_0}$

When substituting it above, we can write

$2f = r_0(1 + \cos\theta_0) = \frac{d}{2\sin\theta_0}(1 + \cos\theta_0) \Rightarrow f = \frac{d}{4}\frac{1+\cos\theta_0}{\sin\theta_0}$

Using the trigonometric identity (See Appendix VI-1)

$$\cot\left(\frac{\alpha}{2}\right) = \frac{1+\cos\alpha}{\sin\alpha}$$

reduces the above equation to

$$f = \frac{d}{4}\cot\left(\frac{\theta_0}{2}\right) \Rightarrow \frac{f}{d} = \frac{1}{4}\cot\left(\frac{\theta_0}{2}\right)$$

15-6. From (15-25) $\Rightarrow \frac{f}{d} = \frac{1}{4}\cot\left(\frac{\theta_0}{2}\right)$

a. $\theta_0 = 90°: \frac{f}{d} = \frac{1}{4}\cot\left(\frac{90°}{2}\right) = \frac{1}{4}\cot(45°) = \frac{1}{4} = 0.25$

$\theta_0 = 180°: \frac{f}{d} = \frac{1}{4}\cot\left(\frac{180°}{2}\right) = \frac{1}{4}\cot(90°) = 0$

$90° < \theta_0 < 180°: \quad 0.25 > f/d > 0$

b. $\theta_0 = 0°: \frac{f}{d} = \frac{1}{4}\cot(0) = \infty$

$\theta_0 = 90°: \frac{f}{d} = \frac{1}{4}\cot\left(\frac{90°}{2}\right) = \frac{1}{4}\cot(45°) = 0.25$

$0° < \theta_0 < 90°: \quad \infty > f/d > 0.25$

15-7. The far-field region is commonly taken to exist a distances greater than $2D^2/\lambda$ from the antenna, $\lambda$ being the wavelength, D: dimension of diameter.

$R \geq 2D^2/\lambda$, $D = 10$ meter, $f = 2$ GHz $\rightarrow \lambda = 0.15$ meters

At 2GHz $\Rightarrow$ $R \geq 2(100)/(0.15) = 1333.3$ meters

$f = 4$ GHz $\rightarrow \lambda = 0.075$ meters.

At 4GHz $\Rightarrow$ $R \geq 2(100)/(0.075) = 2666.66$ meters.

15-8. From (2-110)

$$A_{em} = \frac{\lambda^2}{4\pi} D \Rightarrow D = \frac{4\pi}{\lambda^2} A_{em}$$

Using (12-40)

$$D = \frac{4\pi}{\lambda^2} A_{em} = \frac{4\pi}{\lambda^2} \epsilon_{ap} A_p = \frac{4\pi}{\lambda^2}(1)(\pi a^2) = \frac{4\pi^2}{\lambda^2}\left(\frac{d}{2}\right)^2 = \left(\frac{\pi d}{\lambda}\right)^2$$

15-9. On the surface of the reflector, the current density is given by (15-29), which can be written using (15-32) and (15-32a), as

$$\underline{J}_s = 2\sqrt{\frac{\epsilon}{\mu}}\,[\hat{n}\times(\hat{S}_i\times\underline{E}_i)] = 2\sqrt{\frac{\epsilon}{\mu}}\,C_1\sqrt{G_f(\theta',\phi')}\,\frac{e^{-jkr'}}{r'}\,[\hat{n}\times(\hat{a}_{r'}\times\hat{e}_i)]$$

The polarization of the source can be written as

$$\hat{a}_y = \hat{a}_{r'}\sin\theta'\sin\phi' + \hat{a}_{\theta'}\cos\theta'\sin\phi' + \hat{a}_{\phi'}\cos\phi'$$

Referring to Figure 15.15, the unit vector $\hat{e}_i$ can be written as

$$\hat{e}_i = \frac{\hat{a}_{r'}\times(\hat{a}_y\times\hat{a}_{r'})}{|\hat{a}_{r'}\times(\hat{a}_y\times\hat{a}_{r'})|}$$

which by using the vector identity

$$\underline{A}\times(\underline{B}\times\underline{C}) = (\underline{A}\cdot\underline{C})\underline{B} - (\underline{A}\cdot\underline{B})\underline{C}$$

can be expressed as

$$\hat{e}_i = \frac{(\hat{a}_{r'}\cdot\hat{a}_{r'})\hat{a}_y - (\hat{a}_{r'}\cdot\hat{a}_y)\hat{a}_{r'}}{|(\hat{a}_{r'}\cdot\hat{a}_{r'})\hat{a}_y - (\hat{a}_{r'}\cdot\hat{a}_y)\hat{a}_{r'}|} = \frac{\hat{a}_y - (\hat{a}_{r'}\cdot\hat{a}_y)\hat{a}_{r'}}{|\hat{a}_y - (\hat{a}_{r'}\cdot\hat{a}_y)\hat{a}_{r'}|}$$

Using the transformation of (VII-12) or (VII-12a), we can write the numerator as

$$\hat{a}_y - (\hat{a}_{r'}\cdot\hat{a}_y)\hat{a}_{r'} = \hat{a}_{\theta'}\cos\theta'\sin\phi' + \hat{a}_{\phi'}\cos\phi'$$

and the denominator as

$$|\hat{a}_y - (\hat{a}_{r'}\cdot\hat{a}_y)\hat{a}_{r'}| = \sqrt{1-\sin^2\theta'\sin^2\phi'}$$

Thus $$\hat{e}_i = \frac{\hat{a}_y\cdot(\hat{a}_{r'}\cdot\hat{a}_y)\hat{a}_{r'}}{|\hat{a}_y\cdot(\hat{a}_{r'}\cdot\hat{a}_y)\hat{a}_{r'}|} = \frac{\hat{a}_{\theta'}\cos\theta'\sin\phi' + \hat{a}_{\phi'}\cos\phi'}{\sqrt{1-\sin^2\theta'\cdot\sin^2\phi'}}$$

and $$\underline{J}_s = 2\sqrt{\frac{\epsilon}{\mu}}\,C_1\sqrt{G_f(\theta',\phi')}\,\frac{e^{-jkr'}}{r'}[\hat{n}\times(\hat{a}_{r'}\times\hat{e}_i)] = 2\sqrt{\frac{\epsilon}{\mu}}\cdot C_1\sqrt{G_f(\theta',\phi')}\,\frac{e^{-jkr'}}{r'}\,\underline{\mu}$$

where $\underline{\mu} = \hat{n}\times(\hat{a}_{r'}\times\hat{e}_i) = (\hat{n}\times\hat{e}_i)\hat{a}_{r'} - (\hat{n}\times\hat{a}_{r'})\hat{e}_i$.

Eventually $\underline{\mu}$ reduces to (15-34) by using (15-18) and (VII-12) and (VII-12a)

15-10. $\underline{E} = (\hat{a}_x + \hat{a}_y \sin\phi \cos\phi)\, f(r,\theta,\phi)$

a. Cross Pol $= \sin\phi\cos\phi$

Maximum : $\phi = 0 \Rightarrow$ zero

$\phi = 90° \Rightarrow$ zero

$\phi = 180° \Rightarrow$ zero

b. Cross Pol $= \sin\phi\cos\phi = \frac{1}{2}\sin(2\phi)$

Maximum ; $\phi = 45° \Rightarrow 0.5$

$\phi = 135° \Rightarrow 0.5$

c. $\underline{E} = (\hat{a}_x + \hat{a}_y \sin\phi\cos\phi) f(r,\theta,\phi) = \dfrac{(\hat{a}_x + \hat{a}_y \sin\phi\cos\phi)}{\sqrt{1+\sin^2\phi\cos^2\phi}}\, f(r,\theta,\phi)$

$\underline{E} = \hat{a}_w f(r,\theta,\phi)$, $\hat{a}_w = \dfrac{\hat{a}_x + \hat{a}_y \sin\phi\cos\phi}{\sqrt{1+\sin^2\phi\cos^2\phi}}$

$PLF = |\hat{a}_x \cdot \hat{a}_w^*|^2 = \left.\dfrac{1}{1+\sin^2\phi\cos^2\phi}\right|_{\phi=45°} = \frac{1}{2} = -3\,dB$

d. $PLF = |\hat{a}_y \cdot \hat{a}_w^*|^2 = \left.\dfrac{\sin^2\phi\cos^2\phi}{1+\sin^2\phi\cos^2\phi}\right|_{\phi=45°} = \dfrac{0.25}{1.25} = 0.2 = -7\,dB$

e. $\hat{a}_a = \dfrac{(\hat{a}_x + \hat{a}_y \sin\phi\cos\phi)}{\sqrt{1+\sin^2\phi\cos^2\phi}}$

$PLF = \left|\left(\dfrac{\hat{a}_x + \hat{a}_y \sin\phi\cos\phi}{\sqrt{1+\sin^2\phi\cos^2\phi}}\right)\cdot\left(\dfrac{\hat{a}_x + \hat{a}_y \sin\phi\cos\phi}{\sqrt{1+\sin^2\phi\cos^2\phi}}\right)\right|^2$

$= \left|\dfrac{1+\sin^2\phi\cdot\cos^2\phi}{1+\sin^2\phi\cdot\cos^2\phi}\right|^2 = 1 = 0\,dB$

15-11 $\displaystyle\int_0^{2\pi}\int_0^{\theta_0} \underline{J}_s e^{jk\underline{r}'\cdot\hat{a}_r}\, ds' = 2\sqrt{\frac{\epsilon}{\mu}}\, C_1 \underline{I}$

where $\underline{I} = \displaystyle\int_0^{2\pi}\int_0^{\theta_0} \frac{\sqrt{G_f(\theta',\phi')}}{r'} e^{-jkr'[1-\sin\theta'\sin\theta\cos(\phi'-\phi)-\cos\theta'\cos\theta]}$

$\times\, [\hat{n}\times(-\hat{a}_z\times\hat{e}_r)]\, r'^2 \sin\theta' \sec\left(\frac{\theta'}{2}\right) d\theta' d\phi'$

By using (15-37a)

$\hat{a}_\theta \cdot \underline{M} = \hat{a}_\theta \cdot [-\hat{a}_z(\hat{n}\cdot\hat{e}_r) - \hat{e}_r \cos(\frac{\theta'}{2})] = (\hat{n}\cdot\hat{e}_r)\sin\theta - \hat{a}_\theta\cdot\hat{e}_r\cos(\frac{\theta'}{2})$

$\hat{a}_\phi \cdot \underline{M} = \hat{a}_\phi \cdot [-\hat{a}_z(\hat{n}\cdot\hat{e}_r) - \hat{e}_r\cos(\frac{\theta'}{2})] = -\hat{a}_\phi\cdot\hat{e}_r\cos(\frac{\theta'}{2})$ (Continued)

15-11 (Cont'd)

Toward $\theta=\pi$ equation (15-49) reduces to (15-51). This is accomplished by making the following substitutions. The exponential of (15-49) can be written as

$$r'[1-\sin\theta'\sin\theta\cdot\cos(\phi'-\phi)-\cos\theta'\cdot\cos\theta]_{\theta=\pi} = r'[1+\cos\theta'] = 2f$$

by using (15-14). Also

$$\hat{a}_\theta\cdot\underline{\mu}\,|_{\theta=\pi} = [(\hat{n}\cdot\hat{e}_r)\sin\theta - \hat{a}_\theta\cdot\hat{e}_r\cos(\tfrac{\theta'}{2})]_{\theta=\pi} = -\hat{a}_\theta\cdot\hat{e}_r\cos(\tfrac{\theta'}{2})\,\Big|_{\theta=\pi}$$

$$\hat{a}_\phi\cdot\underline{\mu}\,|_{\theta=\pi} = -\hat{a}_\phi\cdot\hat{e}_r\cos(\tfrac{\theta'}{2})\,|_{\theta=\pi}$$

Assuming azimuthal symmetry, we can write I as

$$I\Big|_{\theta=\pi} = -2\pi\int_0^{\theta_0}\sqrt{G_f(\theta')}\,e^{-jk(2f)}\,(\hat{a}_\theta\cdot\hat{e}_r+\hat{a}_\phi\cdot\hat{e}_r)\Big|_{\theta=\pi} r'\sec(\tfrac{\theta'}{2})\cos(\tfrac{\theta'}{2})\sin(\theta')\,d\theta'$$

$$= -2\pi\int_0^{\theta_0}(\hat{a}_\theta\cdot\hat{e}_r+\hat{a}_\phi\cdot\hat{e}_r)\Big|_{\theta=\pi}\sqrt{G_f(\theta')}\,e^{-j2kf}\,r'\sin\theta'\,d\theta'$$

Since $r' = f\sec^2(\frac{\theta'}{2})$, as given by (15-14a), we can write I as

$$I|_{\theta=\pi} = -2\pi f\cdot e^{-j2kf}\int_0^{\theta_0}[(\hat{a}_\theta\cdot\hat{e}_r+\hat{a}_\phi\cdot\hat{e}_r)]_{\theta=\pi}\cdot\sqrt{G_f(\theta')}\cdot\frac{\sin\theta'}{\cos^2(\theta'/2)}\,d\theta'$$

Since $\sin\theta' = \sin(\frac{\theta'}{2}+\frac{\theta'}{2}) = 2\sin(\frac{\theta'}{2})\cos(\frac{\theta'}{2})$

$$I|_{\theta=\pi} = -4\pi f\,e^{-j2kf}\int_0^{\theta_0}[(\hat{a}_\theta\cdot\hat{e}_r+\hat{a}_\phi\cdot\hat{e}_r)]_{\theta=\pi}\sqrt{G_f(\theta')}\cdot\tan(\tfrac{\theta'}{2})\,d\theta'$$

Assuming the cross polarized field is small

$$(\hat{a}_\theta\cdot\hat{e}_r+\hat{a}_\phi\cdot\hat{e}_r)_{\theta=\pi} \simeq -1$$

Thus $$I|_{\theta=\pi} = -4\pi f\,e^{-j2kf}\int_0^{\theta_0}\sqrt{G_f(\theta')}\tan(\tfrac{\theta'}{2})\,d\theta'$$

and $$E(r,\theta=\pi) = -j\frac{\omega\mu\,e^{-jkr}}{2\pi r}\left[\sqrt{\frac{\epsilon}{\mu}}\cdot\frac{P_t}{2\pi}\right]^{1/2}(4\pi f\,e^{-j2kf})\int_0^{\theta_0}\sqrt{G_f(\theta')}\cdot\tan(\tfrac{\theta'}{2})\,d\theta'$$

$$E(r,\theta=\pi) = -j\frac{2\omega\mu f}{r}\left[\sqrt{\frac{\epsilon}{\mu}}\cdot\frac{P_t}{2\pi}\right]^{1/2}\cdot e^{-jk(r+2f)}\int_0^{\theta_0}\sqrt{G_f(\theta')}\cdot\tan(\tfrac{\theta'}{2})\,d\theta'$$

which is used to form (15-52a), (15-53), and (15-54).

15-12 $f = 3\,GHz$, $\lambda = 0.1\,m$, $d = 1$ meter.

$D_o = \left(\frac{\pi d}{\lambda}\right)^2 \cdot \epsilon_{ap}$ , $\epsilon_{ap}$: aperture efficiency

a. uniform illumination, $\epsilon_{ap} = 1$

$$D_o = \left(\frac{\pi \cdot 1}{0.1}\right)^2 = 986.96 = 29.94\,dB$$

b. $\epsilon_{ap} = \epsilon_t \cdot \epsilon_s = (0.8)(0.85) = 0.68$

$$D_o = \left(\frac{\pi \cdot 1}{0.1}\right)^2 \cdot (0.68) = 671.13$$

$$D_o = 671.13 = 28.27\,dB$$

total aperature efficiency : $\epsilon_{ap} = 0.68 = -1.6749\,dB$

15-13 $f/d = 0.4284$

a. Using (15-24)

$$\theta_o = \tan^{-1}\left[\frac{\frac{1}{2}\left(\frac{f}{d}\right)}{\left(\frac{f}{d}\right)^2 - \frac{1}{16}}\right] = \tan^{-1}\left[\frac{0.5\,(0.4284)}{(0.4284)^2 - 1/16}\right] = \tan^{-1}(1.7698) = 60.53^\circ$$

$$\text{HPBW}_o = 2\theta_o = 121.06^\circ$$

b. Using (15-55)

$$\epsilon_{ap} = \cot^2\left(\frac{\theta_o}{2}\right)\left|\int_0^{\theta_o}\sqrt{G_f(\theta')}\cdot\tan\left(\frac{\theta'}{2}\right)d\theta'\right|^2 = \cot^2\left(\frac{\theta_o}{2}\right)\left|\int_0^{\theta_o}\sqrt{2}\cos\left(\frac{\theta'}{2}\right)\tan\left(\frac{\theta'}{2}\right)d\theta'\right|^2$$

$$= 2\cot^2\left(\frac{\theta_o}{2}\right)\left|\int_0^{\theta_o}\sin\left(\frac{\theta'}{2}\right)d\theta'\right|^2 = 2\cot^2\left(\frac{\theta_o}{2}\right)\left|\left[-2\cos\left(\frac{\theta'}{2}\right)\right]_0^{\theta_o}\right|^2$$

$$\epsilon_{ap} = 2\cot^2\left(\frac{\theta_o}{2}\right)\left[2 - 2\cos\left(\frac{\theta_o}{2}\right)\right]^2 = 8\cot^2\left(\frac{\theta_o}{2}\right)\left[1 - \cos\left(\frac{\theta_o}{2}\right)\right]^2$$

$$\epsilon_{ap}\ (\theta_o = 60.53^\circ) = 8\cot^2\left(\frac{60.53^\circ}{2}\right)\left[1 - \cos\left(\frac{60.53^\circ}{2}\right)\right]^2 = 8\cdot(2.937)(0.018576)$$

$$\epsilon_{ap}\ (\theta_o = 60.53^\circ) = 0.43644 \Rightarrow \epsilon_{ap} = 43.644\,\%$$

c. $D_o = \epsilon_{ap}\left(\frac{\pi d}{\lambda}\right)^2$, $f = 10\,GHz \Rightarrow \lambda = \frac{3\times10^8}{10\times10^9} = 3\times10^{-2} = 0.03$ meters

$d = 42.672$ meters, $D_o = 0.43644\left(\frac{\pi(42.672)}{0.03}\right)^2 = 8.715\times10^6 = 69.4\,dB$

d. Using (15-65)

$$\frac{D}{D_o} \geq \left(1 - \frac{m^2}{2}\right)^2 = \left[1 - \frac{1}{2}\left(\frac{\pi}{16}\right)^2\right]^2 = 0.9618 \Rightarrow D \leq 0.9618\,D_o$$

$$D_o \leq 0.9618\,(8.715\times10^6) = 8.3822\times10^6 = 69.234\,dB$$

15-14. $f/d = 0.38$

a. Using (15-24) $\Rightarrow \theta_0 = \tan^{-1}\left[\frac{0.5(0.38)}{(0.38)^2 - 1/16}\right] = 66.68°$

From Figure 15.20 $G_f = \cos^n\theta' = \cos^2\theta'$

b. For $G_f = \cos^2\theta' \Rightarrow \epsilon_{ap}(\theta_0 = 66.68°) \simeq 0.83$

$f/d = 0.38 = 10\lambda/d \Rightarrow d = 10\lambda/(0.38) = 26.3158\lambda$

$D = \left(\frac{\pi d}{\lambda}\right)^2 \epsilon_{ap} = \left[\frac{\pi(26.3158)\lambda}{\lambda}\right]^2 0.83 = 5672.98 = 37.538\text{dB}$

c. From Figure 15.22, the field strength of the $\cos^2(\theta')$ pattern along $\theta' = \theta_0 = 66.68°$ is 8 dB down from the maximum.

15-15 When a reflector is illuminated by a point source located at its focus, the amplitude distribution on its aperture is not uniform. This is evident by recalling that the energy radiates by a spherical source decreases by $1/r^2$ ($1/\rho^2$ for a cylindrical source). After reflection from the paraboloid the rays are parallel to each other and maintain the same energy density distribution. To derive the amplitude taper on the aperture of a paraboloid, let us refer to Figure 15.10. Assuming initially that the feed source is isotropic, the power radiated within a cone of included angle $d\theta'$ (when rotated about the z-axis through 360°) can be written as

$$dP = C\int_0^{2\pi}(\sin\theta' d\theta')\,d\phi' = 2\pi C\sin\theta' d\theta'$$

where C is a constant. This power must also appear in the aperture ring $dA = 2\pi\rho\, d\rho$ where $\rho' = r'\sin\theta'$. Thus we can write

$$\frac{dP}{dA} = \frac{2\pi C\sin\theta' d\theta'}{2\pi\rho' d\rho'} = \frac{2\pi C\cdot\sin\theta' d\theta'}{2\pi r'\sin\theta' d\rho'} = C\frac{1}{r'}\frac{d\theta'}{d\rho'}$$

Since $d\rho' = r' d\theta'$

$$\frac{dP}{dA} = C\frac{1}{r'}\frac{d\theta'}{r' d\theta'} = \frac{C}{(r')^2},$$ Using (15-14a), it reduces to

$$\frac{dP}{dA} = \frac{C}{f^2\sec^4(\theta/2)} = \frac{C}{[f\cdot\sec^2(\theta'/2)]^2} = \frac{C}{f^2[1+\tan^2(\theta'/2)]^2}$$

by utilizing the trigometric identity of $\sec^2 x = 1+\tan^2 x$ (See Appendix VI)

It can also be written as

$$\frac{dP}{dA} = \frac{C}{f^2[1+\sin^2(\theta'/2)/\cos^2(\theta'/2)]^2} = \frac{C}{f^2}\cos^4\left(\frac{\theta'}{2}\right)$$

If the source is not omnidirectional, and it has a symmetric power pattern $G_f(\theta')$, then $\frac{dP}{dA} = G_f(\theta')\cos^4\left(\frac{\theta'}{2}\right)$. For $\frac{dP}{dA}$ to remain constant, $G_f(\theta') = \sec^4\left(\frac{\theta'}{2}\right)$.

15-16.

a. $\iint_{\Omega} G_f(\theta')\,d\Omega = \int_0^{2\pi}\int_0^{\pi/2} G_0 \cos^4\left(\frac{\theta'}{2}\right)\sin\theta' d\theta' d\phi' = 2\pi G_0 \int_0^{\pi/2} \cos^4\left(\frac{\theta'}{2}\right)\sin\theta' d\theta' = 4\pi$

Thus $G_0\int_0^{\pi/2}\cos^4\left(\frac{\theta'}{2}\right)\sin\theta' d\theta' = G_0\int_0^{\pi/2} 2\cdot\cos^4\left(\frac{\theta'}{2}\right)\sin\left(\frac{\theta'}{2}\right)\cos\left(\frac{\theta'}{2}\right)d\theta' = 2$

or $G_0\int_0^{\pi/2}\cos^5\left(\frac{\theta'}{2}\right)\sin\left(\frac{\theta'}{2}\right)d\theta' = G_0\int_0^{\pi/2}\frac{\cos^5\left(\frac{\theta'}{2}\right)d\left(\cos\left(\frac{\theta'}{2}\right)\right)}{-1/2} = 1$

$G_0\int_0^{\pi/2}\left[\cos\left(\frac{\theta'}{2}\right)\right]^5 d\left[\cos\left(\frac{\theta'}{2}\right)\right] = G_0\left[\frac{\cos^6(\theta'/2)}{6}\right]_0^{\pi/2} = -\frac{1}{2}$

$G_0\left[\cos^6\left(\frac{\pi}{4}\right) - \cos(0)\right] = -3 \Rightarrow G_0 = \frac{3}{1-\cos^6(\pi/4)} = \frac{3}{1-0.125} = 3.4286$

b. $\epsilon_{ap} = \cot^2\left(\frac{\theta_0}{2}\right)\left|\int_0^{\theta_0}\sqrt{G_f(\theta')}\cdot\tan\left(\frac{\theta'}{2}\right)d\theta'\right|^2 = \cot^2\left(\frac{\theta_0}{2}\right)\left|\int_0^{\theta_0}\sqrt{G_0}\cdot\cos^2\left(\frac{\theta'}{2}\right)\cdot\tan\left(\frac{\theta'}{2}\right)d\theta'\right|^2$

$= G_0\cdot\cot^2\left(\frac{\theta_0}{2}\right)\left|\int_0^{\theta_0}\cos\left(\frac{\theta'}{2}\right)\sin\left(\frac{\theta'}{2}\right)d\theta'\right|^2 = G_0\cdot\cot^2\left(\frac{\theta_0}{2}\right)\left|\frac{1}{2}\int_0^{\theta_0}\sin\theta' d\theta'\right|^2$

$= G_0\cot^2\left(\frac{\theta_0}{2}\right)\cdot\frac{1}{4}\cdot\left[-\cos\theta'\Big|_0^{\theta_0}\right]^2 = G_0\cdot\cot^2\left(\frac{\theta_0}{2}\right)\left[\frac{1-\cos\theta_0}{2}\right]^2 = G_0\cot^2\left(\frac{\theta_0}{2}\right)\sin^4\left(\frac{\theta_0}{2}\right)$

$\epsilon_{ap} = G_0\frac{\cos^2\left(\frac{\theta_0}{2}\right)}{\sin^2\left(\frac{\theta_0}{2}\right)}\cdot\sin^4\left(\frac{\theta_0}{2}\right) = G_0\cos^2\left(\frac{\theta_0}{2}\right)\sin^2\left(\frac{\theta_0}{2}\right) = G_0\left[\cos\left(\frac{\theta_0}{2}\right)\cdot\sin\left(\frac{\theta_0}{2}\right)\right]^2$

$\epsilon_{ap} = G_0\left[\frac{1}{2}\sin(\theta_0)\right]^2 = \frac{G_0}{4}\sin^2(\theta_0) = \frac{3.4286}{4}\sin^2\theta_0 = 0.85715\sin^2\theta_0$

$\epsilon_{ap} = 0.85715\sin^2\theta_0$

c. $\theta_0 = 90°$ for maximum aperture efficiency. The total subtended angle is equal to $\Phi_0 = 2\theta_0 = 180°$. The maximum aperture efficiency is equal to $\epsilon_{ap}|_{max} = 85.715\%$

15-17 a. $\theta_0 = \tan^{-1}\left[\frac{0.5\,(0.25)}{(0.25)^2 - 1/16}\right] = 90°$

From (15-59 b) $\epsilon_{ap} = 40\left\{\sin^4(45°) + \ln(\cos(45°))\right\}^2\cot(45°) = 0.3730$

b. $D = 0.373\left[\frac{\pi\cdot 8}{3\times10^8/(5\times10^9)}\right]^2 = 65446.4434$

c. $\epsilon_s = \frac{\int_0^{90°} G_f(\theta')\cdot\sin\theta'\cdot d\theta'}{\int_0^{90°} G_f(\theta')\cdot\sin\theta'\cdot d\theta'} = 1.$ Because the dish has such a large included angle, surrounds the feed pattern completely.

d. $\epsilon_s\epsilon_t = \epsilon_{ap} \Rightarrow \epsilon_t = \frac{\epsilon_{ap}}{\epsilon_s} = \frac{0.373}{1} = 0.373$

15-18 The integral to find $\epsilon_{ap}$ or $\epsilon_t$ directly is difficult.
Best way is to find $\epsilon_{ap}$ from directivity and then divide by $\epsilon_{spillover}$

$$D_o = \left(\frac{\pi d}{\lambda}\right)^2 \epsilon_{ap} = 5.42 \times 10^6$$

$$\left[\frac{\pi(10)}{(3\times10^8/25\times10^9)}\right]^2 \epsilon_{ap} = 5.42\times10^6 \Rightarrow \epsilon_{ap} = 0.7909$$

$$\epsilon_s = \frac{G_o \int_0^{30^\circ} \cos^{10}(\theta')\sin\theta' d\theta'}{G_o \int_0^{90^\circ} \cos^{10}(\theta')\sin\theta' d\theta'} \Rightarrow \int \cos^{10}\theta'\sin\theta' d\theta' = \frac{-\cos^{m+1}\theta'}{m+1}$$

$$\epsilon_s = \frac{-\cos^{11}(\theta')\Big|_0^{30^\circ}}{-\cos^{11}(\theta')\Big|_0^{90^\circ}} = \frac{0.2055-1}{-1} = 0.7945 = \epsilon_s$$

$$\epsilon_t = \frac{\epsilon_{ap}}{\epsilon_s} \quad \text{Since all other efficiencies are 100\%}$$

$$\epsilon_t = \frac{0.7909}{0.7945} = 0.9954 = \epsilon_t$$

15-19 $f = 9\,GHz \Rightarrow \lambda = \frac{3\times10^8}{9\times10^9} = \frac{1}{3}\times10^{-1} = 0.033\dot{3} = \frac{1}{30}$ meter

a. $$\theta_o = \tan^{-1}\left[\frac{\frac{1}{2}\left(\frac{f}{d}\right)}{\left(\frac{f}{d}\right)^2 - \frac{1}{16}}\right] = \tan^{-1}\left[\frac{\frac{1}{2}(0.536)}{(0.536)^2-\frac{1}{16}}\right] = \tan^{-1}\left[\frac{0.268}{(0.536)^2-\frac{1}{16}}\right] = \tan^{-1}(1.1922)$$

$$\theta_o = 50^\circ$$

$$\epsilon_{ap}(n=2) = 24\left\{\sin^2\left(\frac{\theta_o}{2}\right) + \ln\left[\cos\left(\frac{\theta_o}{2}\right)\right]\right\}^2 \cdot \cot^2\left(\frac{\theta_o}{2}\right)$$

$$= 24\left\{\sin^2(25) + \ln[\cos(25^\circ)]\right\}^2 \cot^2(25)$$

$$= 24\left\{(0.4226)^2 + \ln[0.9063]\right\}^2 (2.1445) = 0.7099$$

$$\epsilon_{ap} = 0.71$$

$$D_o = \left(\frac{\pi d}{\lambda}\right)^2 \epsilon_{ap} = \left(\frac{\pi(10)}{1/30}\right)^2 (0.71) = [300\pi]^2(0.71) = (942.478)^2\,0.71$$

$$D_o = 8.882644\times10^5\,(0.71) = 6.306677\times10^5 = 57.998 \simeq 58\,dB$$

b. $$A_{em} = \frac{\lambda^2}{4\pi} D_o = \frac{1}{4\pi}\left(\frac{1}{30}\right)^2 (6.306677\times10^5) = 55.763\ m^2$$

$$= \epsilon_{ap} A_{physical} = 0.71\left(\frac{\pi d^2}{4}\right) = 0.71\left(\frac{\pi\cdot100}{4}\right) = 55.763\ m^2$$

$$A_{em} = \frac{P_T}{W_i} \Rightarrow P_T = A_{em} W_i = 55.763\cdot(10\times10^{-6})$$

$$P_T = 557.633\times10^{-6} = 0.557633\times10^{-3}\ watts$$

$$P_T = 557.633\ \mu watts = 0.557633\ m\,watts$$

$$A_{phys} = \pi r^2 = \pi\left(\frac{d}{2}\right)^2 = \frac{\pi d^2}{4} = \frac{\pi\cdot(10)^2}{4} = 78.5398\ m^2$$

15-20

a. $D \simeq \varepsilon_{ap}\left(\frac{\pi d}{\lambda}\right)^2$ , $\lambda = \frac{3\times10^8}{5\times10^9} = 0.06\ m$

$C = 2\pi a = 2\pi\left(\frac{d}{2}\right) = \pi d = 3\pi$ meters

$D \simeq \varepsilon_{ap}\left(\frac{\pi d}{\lambda}\right)^2 = \varepsilon_{ap}\left(\frac{C}{\lambda}\right)^2 = 0.75\left(\frac{3\pi}{0.06}\right)^2 = 0.75\left(\frac{3\pi}{3/50}\right)^2 = 0.75\,(50\pi)^2$

$D \simeq 0.75\cdot(2500\pi^2) \simeq 18,505.51 \simeq 42.673\ dB$

b. $A_{em} = \frac{\lambda^2}{4\pi} D_o = \frac{(0.06)^2}{4\pi}(18,505.51) = 5.3\ m^2$

$P_T = A_{em}\, W_i = 5.3\,(10\times10^{-6}) = 53\times10^{-6}$ watts

c. $P_T = 53\times10^{-6}\,(1-|\Gamma|^2) = 53\times10^{-6}\,(1-|0.2|^2) = 50.88\times10^{-6}$ Watts

$P_T = 50.88\times10^{-6}$ watts

15-21 $\theta_o = 60^\circ$, $\lambda = \frac{30\times10^9}{10\times10^9} = 3\ cm = 0.03m$

$\frac{f}{d} = \frac{1}{4}\cot\left(\frac{\theta_o}{2}\right) = \frac{1}{4}\cot\left(\frac{60^\circ}{2}\right) = \frac{1}{4}\cot(30^\circ) = \frac{1}{4}(1.732) = 0.433$

$d = f/0.433 = 5/0.433 = 11.547$ meters

$d = 11.547/0.03 = 384.9\lambda \Rightarrow a = 192.45\lambda$

a. $HPBW \simeq \frac{29.2}{a/\lambda} = \frac{29.2}{192.45} = 0.15^\circ$

b. Sidelobe Level $= -17.6$ dB

c. Directivity $= \left(\frac{\pi d}{\lambda}\right)^2 = [\pi(384.9)]^2 = (1,209.199)^2 = 1,462,162.253$

$D_o = 1,462,162.253 = 61.65$ dB

d. $D_o$ (kraus) $= \frac{41.253}{0.15(0.15)} = 1,833,466.67 = 62.63$ dB

$D_o$ (Tai & Pereira) $= \frac{72,815}{\Theta_{1d}^2 + \Theta_{2d}^2} = \frac{72,815}{2(\Theta_d^2)} = \frac{72,815}{2(0.15)^2} = 1,618,111.11$

$= 62.09$ dB

e. $e^{-(4\pi\sigma/\lambda)^2}$ = loss Factor

$\sigma = 0.64\times10^{-3}\ m / 0.03\ m = 21.33\times10^{-3}\lambda$

$LF = e^{-(4\pi(21.33)\times10^{-3})^2} = e^{-(0.268)^2} = e^{-0.07187}$

$LF(dB) = 20\log_{10}e^{-0.07187} = 20(-0.07187)\log_{10}e$

$= 20\,(-0.07187)\,(0.43429) = (-0.07187)(8.686)$

$LF(dB) = 0.624$ dB

15-22 $f/d = 0.357$, $d = 10$ meters, $G_f(\theta') = \cos^2(\theta'/2)$

a. $\epsilon_s = \dfrac{\int_0^{\theta_0} G_f(\theta')\sin\theta' d\theta'}{\int_0^{\pi} G_f(\theta')\sin\theta' d\theta'} = \dfrac{\int_0^{70^\circ} \cos^2(\theta'/2)\sin\theta' d\theta'}{\int_0^{\pi} \cos^2(\theta'/2)\sin\theta' d\theta'}$

$\theta_0 = \tan^{-1}\left[\dfrac{\frac{1}{2}(f/d)}{(f/d)^2 - (\frac{1}{16})}\right] = \tan^{-1}\left[\dfrac{\frac{1}{2}(0.357)}{(0.357)^2 - 1/16}\right] = \tan^{-1}[2.7483] = 70^\circ$

$N = \int_0^{\theta_0} \cos^2(\theta'/2)\sin\theta' d\theta' = \int_0^{\theta_0} \left(\frac{1+\cos\theta'}{2}\right)\sin\theta' d\theta' = \frac{1}{2}\left\{\int_0^{\theta_0} (\sin\theta' + \cos\theta'\sin\theta')\, d\theta'\right\}$

$= \frac{1}{2}\left\{\left[-\cos\theta' - \frac{\cos^2\theta'}{2}\right]_0^{\theta_0}\right\} = -\frac{1}{2}\left[\cos\theta_0 + \frac{1}{2}\cos^2\theta_0 - 1 - \frac{1}{2}\right]_{\theta_0 = 70^\circ}$

$N = -\frac{1}{2}[0.3420 + \frac{1}{2}(0.1170) - 1.5] = -\frac{1}{2}(-1.0995) = 0.5498$

$D = \int_0^{\pi/2} \cos^2(\theta'/2)\sin\theta' d\theta' = -\frac{1}{2}\left[\cos\theta_0 + \frac{1}{2}\cos^2\theta_0 - 1.5\right]_{\theta_0 = 90^\circ} = \frac{1.5}{2} = 0.75$

$\epsilon_s = \dfrac{N}{D} = \dfrac{0.5498}{0.75} = 0.7330 = 73.30\%$

b. $\epsilon_t = 2\cot^2\left(\frac{\theta_0}{2}\right) \dfrac{\left|\int_0^{\theta_0} \sqrt{G_f(\theta')}\cdot\tan\left(\frac{\theta'}{2}\right) d\theta'\right|^2}{\int_0^{\theta_0} G_f(\theta')\sin\theta' d\theta'} = 2\cot^2\left(\frac{\theta_0}{2}\right)\frac{N}{D}$

$N = \left|\int_0^{\theta_0} \sqrt{G_f(\theta')}\tan\left(\frac{\theta'}{2}\right) d\theta'\right|^2 = \left|\int_0^{70^\circ} \sqrt{\cos^2\left(\frac{\theta'}{2}\right)}\tan\left(\frac{\theta'}{2}\right) d\theta'\right|^2$

$= \left|\int_0^{70^\circ} \cos\left(\frac{\theta'}{2}\right)\cdot\tan\left(\frac{\theta'}{2}\right) d\theta'\right|^2 = \left|\int_0^{70^\circ} \cos\left(\frac{\theta'}{2}\right)\frac{\sin\left(\frac{\theta'}{2}\right)}{\cos\left(\frac{\theta'}{2}\right)} d\theta'\right|^2 = \left|\int_0^{70^\circ} \sin\left(\frac{\theta'}{2}\right) d\theta'\right|^2$

$= \left|2\int_0^{70^\circ} \sin\left(\frac{\theta'}{2}\right) d\left(\frac{\theta'}{2}\right)\right|^2 = \left|2\left[-\cos\left(\frac{\theta'}{2}\right)\right]_0^{70^\circ}\right|^2 = \left|2[-\cos(35^\circ) + 1]\right|^2 = (0.3617)^2$

$N = 0.1308$

$D = \int_0^{\theta_0} G_f(\theta')\sin(\theta') d\theta' = \int_0^{70^\circ} \cos^2\left(\frac{\theta'}{2}\right)\cdot\sin\theta' d\theta' = 0.5498$

$\epsilon_t = 2\cot^2\left(\frac{\theta_0}{2}\right)\cdot\frac{N}{D} = 2\cdot\cot^2(35^\circ)\cdot\frac{0.1308}{0.5498} = 4.0792\left(\frac{0.1308}{0.5498}\right) = 0.9706 = 97.06\%$

c. $\epsilon_{ap} = \epsilon_s \cdot \epsilon_t = 0.7330\,(0.9706) = 0.7115 = 71.15\%$

d. $D_o = \left(\frac{\pi d}{\lambda}\right)^2 \epsilon_{ap}$, $\lambda = \frac{3\times10^8}{10\times10^9} = 0.03$ m, $D_o = \left(\frac{\pi(10)}{0.03}\right)^2\cdot(0.7115) = 780.23\times10^3$

$D_o = 780.23\times10^3 = 58.92$ dB

e. $D = D_o e^{-(4\pi\sigma/\lambda)^2} = 780.23\times10^3\, e^{-(4\pi/100)^2} = 780.23\times10^3\cdot e^{-(0.1257)^2}$

$= 780.23\times10^3\,(0.9843) = 767.98\times10^3 = 58.85$ dB

15-23. *From Figure 15-27.*

a. $f/d = 0.50 \Rightarrow a_1 = 1.34\lambda, b_1 = 0.94\lambda$ with $\rho_1$ and $\rho_2$ of any value.

b. $f/d = 0.75 \Rightarrow a_1 = 1.77\lambda\ (\rho_2 = 2\lambda),\ 1.83\lambda\ (\rho_2 = 3\lambda, 4\lambda, \infty)$

$\Rightarrow b_1 = 1.22\lambda\ (\rho_1 = 2\lambda),\ 1.26\lambda\ (\rho_1 = 3\lambda, 4\lambda, \infty)$

c. $f/d = 1.0 \Rightarrow a_1 = 2.14\lambda\ (\rho_2 = 2\lambda),\ 2.27\ (\rho_2 = 3\lambda),\ 2.37\lambda\ (\rho_2 = 4\lambda),$
$2.4\lambda\ (\rho_2 = \infty)$

$\Rightarrow b_1 = 1.49\lambda\ (\rho_1 = 2\lambda),\ 1.57\lambda\ (\rho_1 = 3\lambda),\ 1.64\lambda\ (\rho_2 = 4\lambda, \infty)$